Lecture 4: From boundary to interior

Today we discuss 2 ways to construct conformal maps between regions using the boundary to guide the construction of the map. (Then we prove Carathéodory's theorem from Lecture 3.)

I. Constructing holomorphic maps onto D_1

Theorem A Let U be a bounded region, and $f \in C^0(\bar{U})$ a nonconstant function with $f|_U$ holomorphic. Then

$$f(\partial U) \subset S^1 \implies f(U) = D_1.$$

(If we assume also that f is 1-to-1, then $f|_U$ is a conformal isomorphism.)

Remark // Of course, by RMT / Carathéodory we can extend this to the case where D_1 is replaced by a region bounded by a C^0 Jordan curve. //

Proof: ($f(U) \subseteq D_1$) By hypothesis, $|f(z)| = 1$ ($\forall z \in \partial U$), so MMP $\Rightarrow$ $|f(z)| \leq 1$ ($\forall z \in \overline{U}$). Given $z_0 \in U$, if $|f(z_0)| = 1$ then $|f(z_0)| \geq |f(z)|$ ($\forall z \in U$) $\xRightarrow[\text{MMP}]{}$ f constant. (contradiction)

So $|f(z_0)| < 1$ ($\forall z_0 \in U$), i.e. $f(U) \subset D_1$.

($f(U) \supseteq D_1$) Given $\alpha \in D_1 \setminus f(U)$, $\varphi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ sends α (and no other point) to 0. So image$(\varphi_\alpha \circ f) \not\ni \{0\}$, and $\frac{1}{\varphi_\alpha \circ f} \in \mathcal{H}ol(U) \cap C^0(\overline{U})$. Since f maps ∂U into S^1, and φ_α maps S^1 into S^1, and $z \mapsto \frac{1}{z}$ maps S^1 into S^1, $\left(\frac{1}{\varphi_\alpha \circ f}\right)(\partial U) \subset S^1 \implies \left|\left(\frac{1}{\varphi_\alpha \circ f}\right)(z)\right| = 1$ ($\forall z \in \partial U$).

As f maps U into D_1, which φ_α maps into D_1, which $z \mapsto \frac{1}{z}$ maps into $(\overline{D}_1)^c$, $\left|\left(\frac{1}{\varphi_\alpha \circ f}\right)(z)\right| > 1$ ($\forall z \in U$)

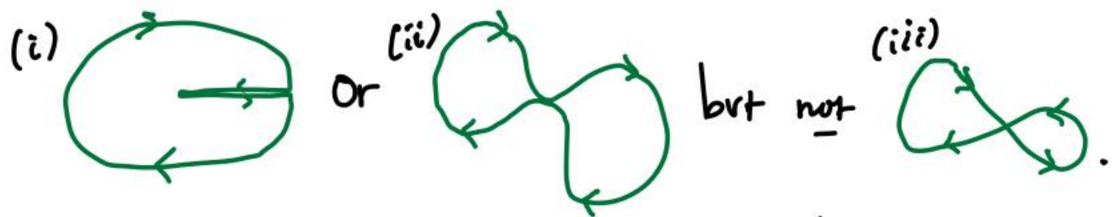
This contradicts MMP. $\square$

This is great, but we'd like a result that gives us that f is 1-to-1 (rather than having to check it). The next result will say that, provided you know that f is holomorphic on a region U' enclosing $\overline{U}$, and ∂U is a Jordan curve, then f is indeed 1-to-1 on U.

II. Constructing biholomorphisms onto D_1

Theorem B Let U' be a region, and $\gamma \subset U'$ a piecewise C^1 Jordan curve, homologous to 0 in U'. Assume given $f \in \mathcal{Hol}(U')$ nonconstant, such that $f\circ\gamma$ is Jordan and disjoint from† $f(\mathrm{Int}(\gamma))$. Then f restricts to a conformal isomorphism from $\mathrm{Int}(\gamma)$ to $\mathrm{Int}(f\circ\gamma)$.

Remark // The hypothesis that γ and $f\circ\gamma$ are Jordan can be weakened a bit. First, both must "have an interior":



(The winding # about every point $\in \mathbb{C}\setminus\gamma$ must be 0 or 1, so not -1.) Moreover, the interior needs to be connected, which means we can't have picture (ii) either. So only (i) is allowed, but that gets us some mileage. //

$\dagger$ recall $\mathrm{Int}(\gamma) := \{\alpha \in \mathbb{C} \mid W(\gamma, \alpha) = 1\}$

Proof: ($f|_{Int(\gamma)}$ 1-1) $\alpha \in Int(\gamma)$ (+ the disjointness hypothesis †) $\Rightarrow$

$f\circ\gamma$ has interior

$$W(f\circ\gamma, f(\alpha)) = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{dw}{w - f(\alpha)} = \frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)-f(\alpha)}\,dz \geq 1 \Rightarrow$$

(argument principle: $f(z) - f(\alpha)$ has a zero [at α] in $Int(\gamma)$)

$W(f\circ\gamma, f(\alpha)) \overset{(*)}{=} 1$ exactly $\Rightarrow$

$f(z) - f(\alpha)$ has one zero in $Int(\gamma)$.

($f(Int(\gamma)) \subset Int(f\circ\gamma)$) follows from $(*)$

($f(Int(\gamma)) \supset Int(f\circ\gamma)$) $\left.\begin{array}{l} f \text{ nonconstant} \\ Int(\gamma) \text{ open} \end{array}\right\} \overset{OMT}{\Longrightarrow} f(Int(\gamma))$ open

$\Rightarrow$ sufficient to show that $f(Int(\gamma))$ is closed in $Int(f\circ\gamma)$.

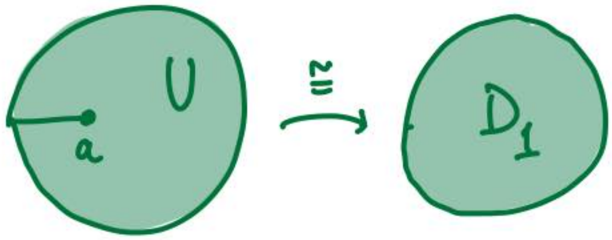
Let $\{z_n\} \subset Int(\gamma)$ be such that $f(z_n) \to \beta \in Int(f\circ\gamma)$.
As $\gamma \cup Int(\gamma)$ is compact, $z_{n_k} \to \alpha \in \gamma \cup Int(\gamma)$
$\overset{f\ C^0}{\Longrightarrow} f(\alpha) = \beta$. If $\alpha \in \gamma$ then $\beta \in f\circ\gamma$, contradicting
$\beta \in Int(f\circ\gamma)$. So $\alpha \in Int(\gamma)$ & thus $\beta \in f(Int(\gamma))$. $\square$

III. Examples

Here are a couple of quick and easy applications (sketches only):

† disjointness hypothesis guarantees $f(\alpha) \notin f\circ\gamma$ so the $\int$ makes sense.

Example 1 // Verify that



is given by $f(z) := \dfrac{\sqrt{g(z)} - i}{\sqrt{g(z)} + i}$, where $g(z) = \left(\dfrac{-i(z+1)}{b(z-1)}\right)^2 + 1$,

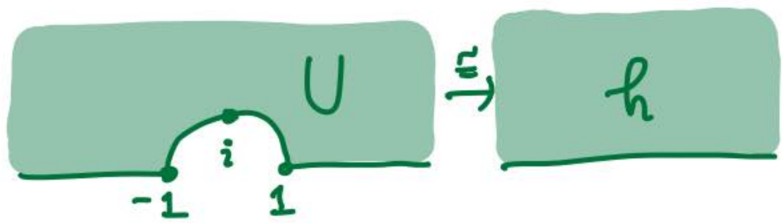
$b := \dfrac{1+a}{1-a}$:

For $z \in (-1, a]$, $g(z) \in \mathbb{R}_{<1}$; while for $z \in S^1$,

$g(z) \in \mathbb{R}_{\geq 1}$. Conclude $f(U) = D_1$ by Theorem A.

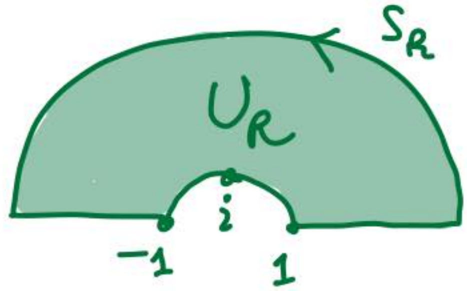
(You can check 1-1 if you want.) //

Example 2 // Verify that



is given by $f(z) := z + \frac{1}{z}$:

Consider the family of regions



Clearly "$f(S_R) \to \infty$" as $R \to \infty$. Furthermore,

- $z = e^{i\theta}$ $(\theta \in [0, \pi])$ $\Rightarrow z + \frac{1}{z} = 2\cos\theta \in [-2, 2]$

• $z = r \in (-\infty, 1] \cup [1, \infty) \Rightarrow z + \frac{1}{z} = r + \frac{1}{r} = f(r)$, and

$$f'(r) = 1 - \frac{1}{r^2} > 0 \quad (\text{for } |r| > 1)$$

By Theorem B, $U_R = \operatorname{Int}(\partial U_R) \xrightarrow[f]{\cong} \operatorname{Int}(f \circ \partial U_R)$

$R \to \infty$

$$U \xrightarrow[f]{\cong} h.$$

provided only for reference

IV. Caratheodory's Theorem (proof)

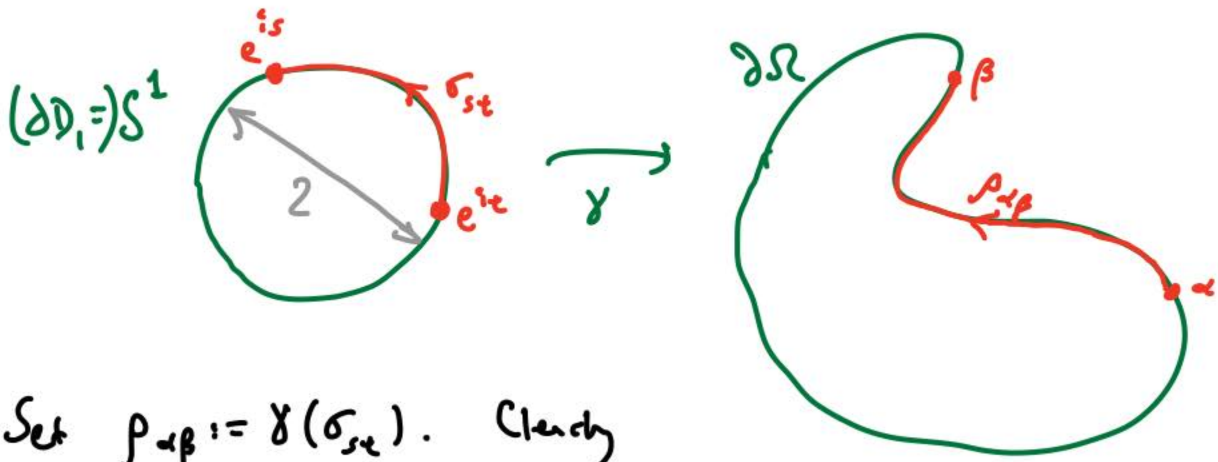
We first shall restrict to the case where the boundary of Ω is a continuous Jordan curve : i.e. there is a C^0 map

unit circle in $\mathbb{C}$

$$\gamma : S^1 \longrightarrow \partial\Omega$$

that is 1-1 & onto, hence a homeomorphism. (Ω is taken to be the bounded component of $\mathbb{C} \setminus \gamma(S^1)$, and is a bounded, simply connected region.)

Clearly γ has a C^0 inverse γ^{-1}. Picking $\delta_0 > 0$ s.t.
$|\underbrace{\gamma(e^{it})}_{\alpha} - \underbrace{\gamma(e^{is})}_{\beta}| \leq \delta_0 \Rightarrow |e^{it} - e^{is}| < 2$, we may define the "shorter arc" σ_{st} on S^1 connecting e^{it}, e^{is} :



Set $\rho_{\alpha\beta} := \gamma(\sigma_{st})$. Clearly

$$\begin{aligned} |\alpha-\beta| \to 0 \quad &\underset{\gamma^{-1}\ C^0}{\Longrightarrow} \quad |e^{it}-e^{is}| \to 0 \\ &\Longrightarrow \quad \mathrm{diam}(\sigma_{st}) \to 0 \\ &\underset{\gamma\ C^0}{\Longrightarrow} \quad \mathrm{diam}(\rho_{\alpha\beta}) \to 0 , \end{aligned}$$

and this convergence is <u>uniform</u> as S^1, $\partial\Omega$ are compact. So for $\delta \in (0, \delta_0)$, setting

$$\eta(\delta) := \sup_{\left(\substack{\alpha,\beta\in\partial\Omega \\ |\alpha-\beta|<\delta}\right)} \{\mathrm{diam}(\rho_{\alpha\beta})\} ,$$

we have $\delta \to 0 \implies \eta(\delta) \to 0$. Taking $\delta_\gamma \in (0, \delta_0)$ s.t. $\eta(\delta_\gamma) < \frac{1}{2}\mathrm{diam}(\partial\Omega)$, $|\alpha-\beta| < \delta_\gamma \Rightarrow \rho_{\alpha\beta}$ is the only arc of $\partial\Omega$ from α to β with diameter $< \eta(\delta_\gamma)$. The existence of δ_γ and this "shortest arc from α to β on $\partial\Omega$" will be used below.

We write D for D_1 (unit disk) in what follows.

$\boxed{\text{Theorem}}$ Let $\varphi : D \to \Omega$ be a conformal isomorphism, with Ω as above (bounded, simply-connected region, with $\partial\Omega$ C^0 Jordan). Then there exists $\hat{\varphi} : \overline{D} \to \overline{\Omega}$ C^0 & 1-to-1, such that $\hat{\varphi}\big|_D = \varphi$ — that is, φ admits an extension to a homeomorphism of the (compact) closures.

Proof : (3 steps, of which the first is the hardest)

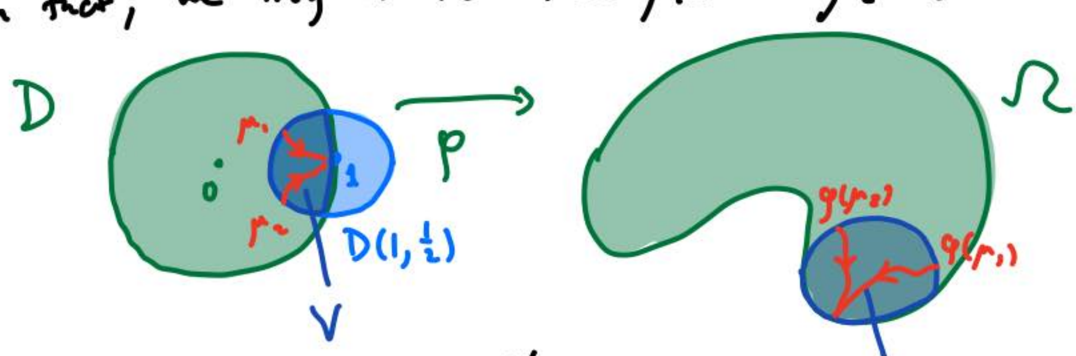
$\boxed{\text{STEP 1}}$ Prove the

Claim: Given $\mu_k : [0,1] \to \overline{D}$ C^0, with $\mu_1(1) = \mu_2(1) \in \partial D$,

$(k=1,2)$ $\quad [0,1) \to D$ (with $[0,1) \subset [0,1]$ and $D \subset \overline{D}$)

we have $\lim_{t\to 1^-} \varphi(\mu_1(t)) = \lim_{t \to 1^-} \varphi(\mu_2(t))$ (the limits exist and are equal).

In fact, we may as well take $\mu_1(1) = 1 = \mu_2(1)$.



Introduce polar coords. on V: $r \in (0, \frac{1}{2})$, $\theta \in (-\theta_0(r), \theta_0(r))$; $\gamma_r : (-\theta_0(r), \theta_0(r)) \to V$

$$\theta \longmapsto 1 - re^{i\theta}.$$

Now since $\varphi(V) \subset \Omega$ = bounded region ($\Rightarrow$ finite area),

$$\infty > A(\varphi(V)) = \int_V |\varphi'|^2 \, dA$$

$$= \int_0^{1/2} \int_{-\theta_0(r)}^{\theta_0(r)} |\varphi'(1-re^{i\theta})|^2 \, r \, d\theta \, dr$$

$$\geq \int_0^{1/2} \left\{ \int_{-\theta_0(r)}^{\theta_0(r)} |\varphi'(1-re^{i\theta})|^2 \, r \, d\theta \right\} \overbrace{\left\{ \int_{-\theta_0(r)}^{\theta_0(r)} r \, d\theta \right\} \frac{1}{\pi r}}^{<1} dr$$

$$\overset{\dagger}{\geq} \int_0^{1/2} \left(\int_{-\theta_0(r)}^{\theta_0(r)} |\varphi'(1-re^{i\theta})| \, r \, d\theta \right)^2 \frac{1}{\pi r} \, dr$$

† Cauchy-Schwarz

$$= \frac{1}{\pi} \int_0^{1/2} \left(\int_{-\theta_0(r)}^{\theta_0(r)} |(\varphi \circ \gamma_r)'(\theta)| \, d\theta \right)^2 d\log(r)$$

$$=: \frac{1}{\pi} \int_0^{1/2} (\ell_r)^2 \, d\log(r), \quad \text{where}$$

ℓ_r = length of $(\varphi \circ \gamma_r)$ is clearly forced to be $< \infty$ (at least for a.e. $r \in (0, \frac{1}{2})$).

Since $d\log(r)$ is nonintegrable at 0, $\exists \, r_j \to 0^+$ s.t. $\ell_{r_j} \to 0^+$. In particular, since these $\ell_{r_j} < \infty$, the limits

$$L_j^{\pm} := \lim_{\theta \to \pm\theta_0(r_j)^{(\mp)}} \varphi(1 - r_j e^{i\theta})$$

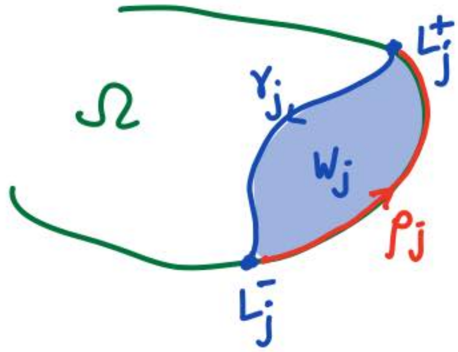
exist. [Proof: taking $\delta > 0$ s.t. $\int_{\theta_0(r_j)-\delta}^{\theta_0(r_j)} |\varphi'(1-r_je^{i\theta})| r_j \, d\theta < \underset{(\text{given})}{\epsilon}$, we see that for $\theta_1, \theta_2 \in (\theta_0 - \delta, \theta_0)$, $|\varphi(1-r_je^{i\theta_1}) - \varphi(1-r_je^{i\theta_2})| < \epsilon$. This is sufficient for the limit to exist.]

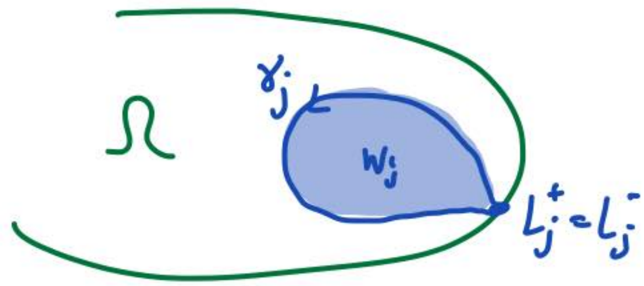
† $\left(\int_a^b |f(x)||g(x)| \, dx \right)^2 \leq \left\{ \int_a^b |f(x)|^2 dx \right\} \left\{ \int_a^b |g(x)|^2 dx \right\}$;

Then put $g(x) \equiv 1$, "dx" $= r d\theta$.

Further, by the compactness of $\overline{\Omega}$ and §1 of Lecture 3, we have $L_j^\pm \in \partial\Omega$. For each j, there are two possible cases:

(I) $\underline{L_j^+ \neq L_j^-}$ (II) $\underline{L_j^+ = L_j^-}$





In case (I), for j suff. large, $\operatorname{diam}(\gamma_j := \varphi \circ \gamma_{r_j}) \leq d_{r_j} < \delta_\gamma$ and so we can take $\rho_j := \rho_{L_j^+ L_j^-}$ ("smaller arc"). We put
$\Gamma_j := \gamma_j \cup \rho_j$.

In case (II), put $\Gamma_j := \gamma_j \cup \{L_j^\pm\}$.

Either way, φ 1-to-1 $\Rightarrow$ Γ_j is a Jordan curve, hence bounds a (bounded) region W_j. (Of course, here we are invoking the Jordan curve theorem.) Setting

$$V_j := \{ 1 - re^{i\theta} \mid r \in (0, r_j),\ |\theta| < \theta_0(r) \},$$

$\partial(\varphi(V_j)) \cap \Omega$ must be γ_j and so we have two cases <u>independent</u> of (I) vs. (II):

(A) $\varphi(V_j) = W_j$

(B) $\varphi(V_j) = \Omega \setminus \overline{W}_j \cap \Omega$

Suppose (B) holds for $j \gg 0$: then $\varphi(D \setminus \overline{V_j}) \subseteq W_j \Rightarrow$

$$A(W_j) \geq A(\varphi(D\setminus\overline{V_j})) = A(\Omega) - A(\varphi(V_j))$$
$$= A(\Omega) - \int_{V_j} |\varphi'|^2 \, dA$$
$$\xrightarrow[(j\to\infty)]{} A(\Omega) \; (\neq 0).$$

Moreover, $|L_j^+ - L_j^-| \leq \ell_{r_j} \Rightarrow \operatorname{diam}(\rho_j) \leq \eta(\ell_{r_j}) \xrightarrow[(j\to\infty)]{} 0$

and $D(L_j^+, \ell_{r_j} + \eta(\ell_{r_j})) \supseteq \Gamma_j$, hence W_j.

$\Rightarrow A(W_j) \leq \pi(\ell_{r_j} + \eta(\ell_{r_j}))^2 \xrightarrow[(j\to\infty)]{} 0$, a contradiction.

So (A) it is, and the argument also shows

$\operatorname{diam} W_j \to 0$, $A(W_j) \to 0$; together with the nesting of the $\{W_j\}$, this implies $\bigcap_j \overline{W_j}$ is a single point Q.

Now let $\epsilon > 0$, $\begin{cases} J \text{ be such that } \operatorname{diam}(W_J) < \epsilon \\ \delta \text{ be such that } t \in (1-\delta, 1) \Rightarrow |\mu_k(t) - 1| < r_J \\ \hfill (k=1,2). \end{cases}$

Then $\mu_k(t) \in V_J \quad (t \in (1-\delta, 1), \; k = 1, 2)$

$\Rightarrow \varphi(\mu_k(t)) \in W_J \quad (\quad " \quad)$

$\Rightarrow |\varphi(\mu_1(t)) - \varphi(\mu_2(t))| < \epsilon.$

Hence, $\lim_{t\to 1^-} \varphi(\mu_1(t)) = \lim_{t\to 1^-} \varphi(\mu_2(t)) = Q$,

proving the Claim.

$\boxed{\text{STEP 2}}$ The continuous extension

Given $P \in \partial D$, if we choose any C^0 path $\mu : [0,1] \to \overline{D}$ with $\mu(1) = P$ and $\mu([0,1)) \subset D$, then (by the Claim) $\hat{\varphi}(P) := \lim_{t \to 1^-} \varphi(\mu(t))$ exists and is independent of the choice of μ.

Given a sequence $\{P_n\} \subset \partial D$ with limit P, we can choose a subsequence $P_{n_j} \in \overline{V_j} \cap \partial D$, with associated paths μ_j having tails in $\overline{V_j}$, hence $\varphi(\mu_j(0,1)) \cap W_j \neq \phi$. An $\epsilon/3$ argument then shows that $\hat{\varphi}(P_j) \longrightarrow \hat{\varphi}(P)$.

$\boxed{\text{STEP 3}}$ Showing $\hat{\varphi}$ is 1-to-1

<u>Lemma</u>: $F : \overline{D} \to \mathbb{C}$ $\begin{cases} C^0 \text{ on } \overline{D} \\ \text{holo. on } D \end{cases}$, $\gamma \subset \partial D$ open arc, and $F|_\gamma \equiv c \implies F \equiv c$.

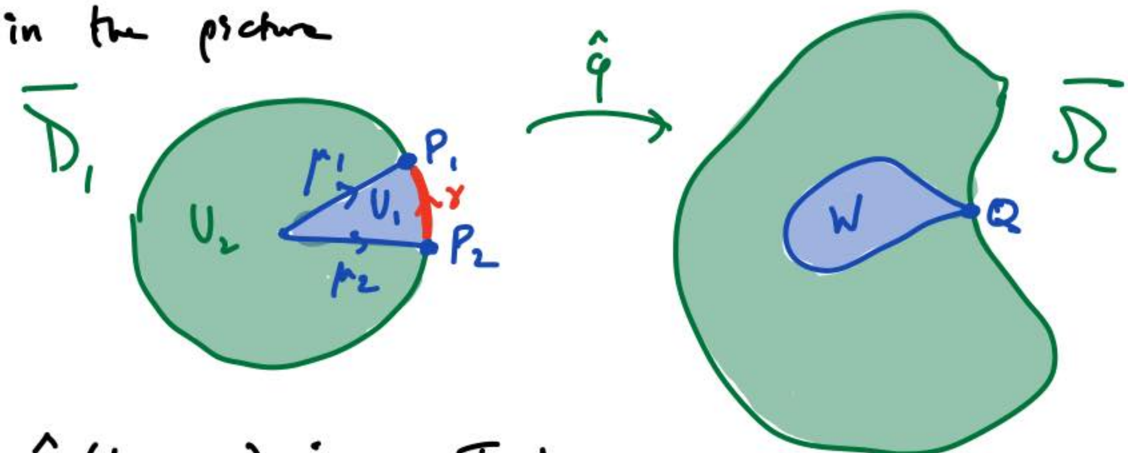
<u>Proof</u>// WOLOG assume $F|_\gamma \equiv 0$. If $F(z_0) \neq 0$ $(z_0 \in D)$ then set $\tilde{F} := F \circ \phi_{z_0}^{-1}$ $(\Rightarrow \tilde{F}(0) \neq 0)$. We then have

$$-\infty < \log|\tilde{F}(0)| \underset{\text{(Jensen)}}{\leq} \frac{1}{2\pi}\int_0^{2\pi} \log|\tilde{F}(re^{i\theta})|\, d\theta$$

for all but countably many values of r where $\tilde{F}$ has a 0 on ∂D_r. Taking $r \to 1^-$, the right-hand side $\to -\infty$ (contradiction). //

Now $\hat{\varphi}(D) \subset \Omega$, $\hat{\varphi}(\partial D) \subset \partial\Omega$, $\hat{\varphi}|_D$ 1-to-1; so it will suffice to check 1-1 on ∂D.

Given $P_1 \neq P_2 \in \partial D$ such that $\hat{\varphi}(P_1) = \hat{\varphi}(P_2)$, in the picture



$\hat{\varphi}(\mu_1 \cup \mu_2)$ is a Jordan curve, bounding a region W. Clearly $\varphi(U_1)$ or $\varphi(U_2) = W$, say $\varphi(U_1)$. Then

$$\hat{\varphi}(\gamma) \subset \overline{W} \cap \partial\Omega = \{Q\} \implies$$

$$\hat{\varphi}|_\gamma \text{ constant } \underset{\text{Lemma}}{\implies} \hat{\varphi} \text{ constant}$$

$$\implies \varphi \text{ constant } \times\!\!\times .$$

$\square$