Lecture 40: The Bergman kernel

Let $\mathcal{D} \subset \mathbb{C}^n$ be a domain, and consider the space of square-integrable holomorphic functions

$$A^2(\mathcal{D}) := \Big\{ f \in \mathcal{H}ol(\mathcal{D}) \,\Big|\, \underbrace{\Big(\int_{\mathcal{D}} |f(\underline{z})|^2 \, dV \Big)^{1/2}}_{=: \|f\|_{A^2(\mathcal{D})}} < \infty \Big\}$$

with inner product

$$\langle f, g \rangle := \int_{\mathcal{D}} f(\underline{z}) \overline{g(\underline{z})} \, dV(\underline{z}) \quad , \quad f, g \in A^2(\mathcal{D}).$$

Last time we proved:

$\boxed{\text{Proposition}}$ With this inner product, $A^2(\mathcal{D})$ is a Hilbert space.

(*) Key ingredient: for $K \subset \mathcal{D}$ compact, $\exists\ C_K$ s.t. $\underbrace{\|f\|_K}_{\text{sup norm}} \le C_K \underbrace{\|f\|_{A^2(\mathcal{D})}}_{L^2 \text{ norm}}$ $(\forall f \in A^2(\mathcal{D}))$

$\boxed{\text{Theorem 1}}$ There exists a unique function

$$K_{\mathcal{D}} : \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{C} \quad (=: \text{Bergman kernel})$$

satisfying

(a) For each fixed $\underline{\zeta} \in \mathcal{D}$, $K_{\mathcal{D}}(\underline{z}, \underline{\zeta}) \in A^2(\mathcal{D})$ (as fcn. of $\underline{z}$)

(b) $K(\underline{z}, \underline{\zeta}) = \overline{K(\underline{\zeta}, \underline{z})}$

(c) $f(\underline{z}) = \int_{\mathcal{D}} K(\underline{z}, \underline{\zeta}) f(\underline{\zeta}) \, dV(\underline{\zeta}) \quad \forall f \in A^2(\mathcal{D})$.

Proof: Observe that from (*) with $K := \{\underline{z}\}$, fixed for the moment

the "evaluation map" $\Phi_{\underline{z}} : A^2(D) \longrightarrow \mathbb{C}$

$$f \longmapsto f(\underline{z})$$

is a continuous linear functional, i.e. $\in A^2(D)^*$. By Riesz, $\exists\, k_{\underline{z}} \in A^2(D)$ s.t. $\Phi_{\underline{z}}(f) = \varphi_{k_{\underline{z}}}(f) \left(= \langle f, k_{\underline{z}} \rangle\right) \forall f \in A^2(D)$, i.e.

$$f(\underline{z}) = \langle f, k_{\underline{z}} \rangle.$$

Set $K(\underline{z}, \underline{\zeta}) := \overline{k_{\underline{z}}(\underline{\zeta})}$. Then

$$\int_D K(\underline{z}, \underline{\zeta}) f(\underline{\zeta})\, dV(\underline{\zeta}) = \langle f, k_{\underline{z}} \rangle = f(\underline{z}) \implies (c).$$

Further,
$$\int_D K(\underline{z}, \underline{t})\, \overline{K(\underline{\zeta}, \underline{t})}\, dV(\underline{t}) \underset{(c)}{=} \overline{K(\underline{\zeta}, \underline{z})}$$
$$\Vert$$
$$\overline{\int K(\underline{\zeta}, \underline{t})\, \overline{K(\underline{z}, \underline{t})}\, dV(\underline{t})} = \overline{\overline{K(\underline{z}, \underline{\zeta})}} = K(\underline{z}, \underline{\zeta})$$

$\implies$ (b). Finally, (a) now follows from $K(\underline{z}, \underline{\zeta}) = \overline{K(\underline{\zeta}, \underline{z})} = k_{\underline{\zeta}}(\underline{z})$.

To see uniqueness, let $K'(\underline{z}, \underline{\zeta})$ be another such function. Then

$$K(\underline{z}, \underline{\zeta}) = \overline{K(\underline{\zeta}, \underline{z})} = \int K'(\underline{z}, \underline{t})\, \overline{K(\underline{\zeta}, \underline{t})}\, dV(\underline{t})$$

$$= \overline{\int K(\underline{\zeta}, \underline{t})\, \overline{K'(\underline{z}, \underline{t})}\, dV(\underline{t})}$$

$$= \overline{\overline{K'(\underline{z}, \underline{\zeta})}} = K'(\underline{z}, \underline{\zeta}). \qquad \square$$

$\boxed{\text{Theorem 2}}$ The Bergman kernel has the invariance property with respect to biholomorphic maps $f: \mathcal{D}_1 \to \mathcal{D}_2$

$$(\det J_f(\underline{z}))(\det \overline{J_f(\underline{\zeta})})K_{\mathcal{D}_2}(f(\underline{z}), f(\underline{\zeta})) = K_{\mathcal{D}_1}(\underline{z}, \underline{\zeta}).$$

Proof: Given $\phi \in A^2(\mathcal{D}_1)$, by uniqueness of $K_{\mathcal{D}_1}$ the following calculation will suffice:

Δ of variable

$$\int_{\mathcal{D}_1} (\det J_f(\underline{z}))(\det \overline{J_f(\underline{\zeta})})K_{\mathcal{D}_2}(f(\underline{z}), f(\underline{\zeta}))\,\phi(\underline{\zeta})\,dV(\underline{\zeta}) =$$

$$\int_{\mathcal{D}_2} (\det J_f(\underline{z}))(\det \overline{J_f(f^{-1}(\underline{\tilde{\zeta}}))})\,K_{\mathcal{D}_2}(f(\underline{z}), \underline{\tilde{\zeta}})\,\phi(f^{-1}(\underline{\tilde{\zeta}}))\underbrace{|\det J_{f^{-1}}(\underline{\tilde{\zeta}})|^2}\,dV(\underline{\tilde{\zeta}}) =$$

view as $|\det J_f(f^{-1}(\underline{\tilde{\zeta}}))|^{-2}$

use fact that the real Jacobian is built out of conformal blocks $\begin{pmatrix} f' & 0 \\ 0 & \overline{f'} \end{pmatrix}$

$$\det J_f(\underline{z}) \int_{\mathcal{D}_2} K_{\mathcal{D}_2}(f(\underline{z}), \underline{\tilde{\zeta}})\,\frac{\phi(f^{-1}(\underline{\tilde{\zeta}}))}{\det J_f(f^{-1}(\underline{\tilde{\zeta}}))}\,dV(\underline{\tilde{\zeta}}) = \cancel{\frac{\det J_f(\underline{z})}{\det J_f(f^{-1}(f(\underline{z})))}}^{\,1}\cdot \phi(f^{-1}(f(\underline{z})))$$

$\in A^2(\mathcal{D}_2)$

$$= \phi(\underline{z}). \qquad \square$$

$\boxed{\text{Theorem 3}}$ Let $\{\phi_j\}_{j=1}^{\infty} \subset A^2(\mathcal{D})$ be a complete orthonormal basis.† Then

$$K_{\mathcal{D}}(\underline{z}, \underline{\zeta}) = \sum_{j=1}^{\infty} \phi_j(\underline{z})\overline{\phi_j(\underline{\zeta})}.$$

$\dagger$ exists b/c

separability of $L^2(\mathcal{D})$

$L^2(\mathcal{D}) \supset A^2(\mathcal{D})$

$\Longrightarrow$ separability of $A^2(\mathcal{D})$.

Proof: The 3 properties are essentially clear: think of (c) as Fourier-expanding $f \in A^2(D)$. All that needs checking is uniform convergence on $K \times K$ (for $K \subset D$ compact).

The idea is

$$\sup_{\underline{z} \in K} \left(\sum |\phi_j(\underline{z})|^2 \right)^{1/2} = \sup_{\underline{z} \in K} \| \{\phi_j(\underline{z})\}_{j=1}^{\infty} \|_{\ell^2} \quad \text{for sequences}$$

i.e. $\|\underline{\phi}\|_{\ell^2} = \sup_{\|\underline{a}\|=1} |\langle \underline{a}, \underline{\phi} \rangle_{\ell^2}|$

$$= \sup_{\substack{\|\{a_j\}\|_{\ell^2} = 1 \\ \underline{z} \in K}} \left| \sum a_j \phi_j(\underline{z}) \right|$$

$$= \sup_{\substack{\|f\|_{A^2(D)} = 1 \\ \underline{z} \in K}} |f(\underline{z})|$$

$$= \sup_{\|f\|_{A^2(D)} = 1} \|f\|_K \leq C_K \quad \text{(by (*))}$$

$$\Rightarrow \sum_{j=1}^{\infty} \left| \phi_j(\underline{z}) \overline{\phi_j(\underline{\zeta})} \right| \underset{\text{Cauchy-Schwarz}}{\leq} \left(\sum_j |\phi_j(\underline{z})|^2 \right)^{1/2} \left(\sum_j |\phi_j(\underline{\zeta})|^2 \right)^{1/2}$$

and the convergence is uniform over $\underline{z}, \underline{\zeta} \in K$. $\square$

Example // $D = B^n$. The functions

$$\{ \underline{z}^{\underline{\alpha}} \}_{\underline{\alpha} \text{ multiindex}} \subset A^2(B^n)$$

are a complete orthogonal system, since

- we can write any $f \in A^2$ as a power series uniquely
- $\int_{B^n} \underline{z}^{\underline{\alpha}} \overline{\underline{z}^{\underline{\beta}}}\, dV = 0$ unless $\underline{\alpha} = \underline{\beta}$ (this boils down to $\int_{\partial D} e^{ia\theta} \overline{e^{ib\theta}}\, d\theta = 0$ unless $a = b$).

Further, writing $\gamma_{\underline{\alpha}} := \int_{B^n} |\underline{z}^{\underline{\alpha}}|^2 dV(\underline{z})$, we have that $\{\gamma_{\underline{\alpha}}^{-1/2} \underline{z}^{\underline{\alpha}}\}$ yield an ortho<u>normal</u> system and

- $K(\underline{z}, \underline{\zeta}) = \sum_{\underline{\alpha}} \gamma_{\underline{\alpha}}^{-1/2} \underline{z}^{\underline{\alpha}} \overline{\underline{\zeta}^{\underline{\alpha}}}$ (using Thm. 3)
- $\gamma_{[k]} = \dfrac{\pi^n k!}{(n+k)!}$ [Exercise]

multiindex $(k, 0, \dots, 0)$

Write $\mathbb{1} = (1, 0, \dots, 0) \in \mathbb{C}^n$, & let $0 < r < 1$.

Then

$$K_{B^n}(\underline{z}, r\mathbb{1}) = \sum_{\underline{\alpha}} \frac{\underline{z}^{\underline{\alpha}} \cdot (r\mathbb{1})^{\underline{\alpha}}}{\gamma_{\underline{\alpha}}} = \sum_{k \geq 0} \frac{z_1^k r^k}{\pi^n k!} (n+k)!$$

$$= \frac{n!}{\pi^n} \sum_{k \geq 0} \underbrace{(rz_1)^k \binom{k+n}{n}}_{\text{or: } (-rz_1)^k \binom{-(n+1)}{k}}$$

$$= \frac{n!}{\pi^n} \cdot \frac{1}{(1 - rz_1)^{n+1}}. \quad (**)$$

More generally, if $\begin{cases} \underline{z} = r\tilde{\underline{z}} \in B^n \text{ where } \|\tilde{\underline{z}}\| = 1 \\ \underline{\zeta} \in B^n \end{cases}$ and $\begin{cases} \rho \in U(n) \\ \rho(\tilde{\underline{z}}) = \mathbb{1} \end{cases}$

Then

$$K_B(\underline{z},\underline{\zeta}) = K_B(r\tilde{\underline{z}},\underline{\zeta}) = K(r\rho^{-1}\mathbb{1},\zeta) \overset{\text{Thm. 2}}{=} K(r\mathbb{1},\rho(\underline{\zeta}))$$

$$= \underbrace{\overline{K(\rho(\underline{\zeta}),r\mathbb{1})}}_{} \underset{\text{by } (**)}{=} \frac{n!}{\pi^n}\cdot\frac{1}{(1-r\overline{(\rho(\underline{\zeta}))_1})^{n+1}}$$

$$= \frac{n!}{\pi^n}\cdot\frac{1}{(1-\underbrace{(r\mathbb{1})\cdot\overline{(\rho(\underline{\zeta}))}}_{})^{n+1}}$$

Hermitian dot product (preserved by $U(n)$)

$z\cdot\overline{w} = \sum z_i \overline{w_i}$.

$$= \frac{n!}{\pi^n}\cdot\frac{1}{(1-(r\rho^{-1}(\mathbb{1}))\cdot\overline{\underline{\zeta}})^{n+1}}$$

$$= \boxed{\frac{n!}{\pi^n}\cdot\frac{1}{(1-z\cdot\overline{\zeta})^{n+1}}}$$

For the unit disk ($n=1$), this is

$$\boxed{K_D(z,\zeta) = \frac{1}{\pi}\cdot\frac{1}{(1-z\overline{\zeta})^2}.}$$

By the invariance result (Theorem 2), if

$$f:\Omega\to D$$

is a mapping as in the RMT, then

$$K_\Omega(z,\zeta) = \frac{f'(z)\overline{f'(\zeta)}}{\pi}\cdot\frac{1}{(1-f(z)\overline{f(\zeta)})^2}.$$

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In a more thorough discussion we would have treated the Bergman metric.

$\boxed{\text{Theorem 4}}$ Every bounded domain has a canonical Hermitian metric (=: Bergman metric). It has negative curvature and is invariant (pulls back to itself) under holomorphic automorphisms.

Proof (partial sketch): $h(z) := \partial\bar\partial \log K_D(z,z)$

$$= \sum_{i,j} \frac{\partial^2}{\partial z_i \partial \bar z_j} \log K_D(z,z)\, dz_i d\bar z_j ,$$

which makes sense b/c $K_D(z,z) = \sum_{j \geq 1} |\phi_j(z)|^2 > 0$ ($\because$ if $= 0$ anywhere then $\exists$ a point z s.t. every $f \in A^2(D)$ vanishes there — absurd).

Meaning of this is that for $\gamma : [0,1] \to D$,

$$\text{length}(\gamma) := \int_0^1 \sqrt{\sum_{i,j} h_{ij}(\gamma(t))\, \gamma_i'(t) \overline{\gamma_j'(t)}}\; dt .$$

To each z and 2-plane $P \subset T_z D$, one assigns a number $K(P)$ =: Gaussian curvature. For the "holomorphic 2-planes" this number is < 0. (We won't have time to prove this.) Invariance under conformal isomorphisms (of $\mathbb{D}$ with itself or with another domain) is a direct consequence of Theorem 2. $\square$

Example// For the unit disk,

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \log \frac{1}{(1-|z|^2)^2} = -2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\log(1-z\bar{z})$$

$$= 2\frac{\partial}{\partial z}\frac{z}{1-z\bar{z}}$$

$$= \frac{2}{(1-|z|^2)^2} \quad [dz\,d\bar{z}, \text{ i.e. } dx^2+dy^2]$$

is the Poincaré metric. It "pulls back" to $\frac{1}{4y^2}(dx^2+dy^2)$ on the upper half-plane (can you show this?). //