Lecture 7 : More on harmonic functions

I. Fourier series

This is a nice application of the Theorem on Dirichlet's problem from Lecture 6. Moreover, it gives a better idea of what the harmonic functions of that Theorem look like.

Let $f \in C^0_{\mathbb{R}}([0,2\pi])$, $f(0) = f(2\pi)$. The Theorem just mentioned guarantees the existence of $u \in C^0(\overline{D_1})$ satisfying:

(a) $u|_{D_1}$ is harmonic, hence of the form

$$u(re^{i\theta}) = \mathrm{Re}\left(\sum_{n\geq 0} \alpha_n (re^{i\theta})^n\right)$$

$a_0 = 2\,\mathrm{Re}\,\alpha_0$
$a_{n>0} = \mathrm{Re}\,\alpha_n$
$b_n = -\mathrm{Im}\,\alpha_n$

$$(*) \qquad = \frac{a_0}{2} + \sum_{n\geq 1} a_n r^n \cos(n\theta) + \sum_{n\geq 1} b_n r^n \sin(n\theta),$$

where the series are absolutely and uniformly convergent on $\overline{D_r}$ for $r<1$

$\&$ (b) $u(e^{i\theta}) = f(\theta)$.

Now using basic trigonometric integrals, (a) gives

$$\frac{1}{\pi}\int_0^{2\pi} \cos(n\theta)\, u(re^{i\theta})\, d\theta = a_n r^n$$

$$\frac{1}{\pi}\int_0^{2\pi} \sin(n\theta)\, u(re^{i\theta})\, d\theta = b_n r^n .$$

Taking $r \to 1^-$ and using (uniform) continuity of u on $\overline{D_1}$ together with (b),

$$\left.\begin{aligned} \frac{1}{\pi}\int_0^{2\pi} \cos(n\theta)\, f(\theta)\, d\theta &= a_n \\ \frac{1}{\pi}\int_0^{2\pi} \sin(n\theta)\, f(\theta)\, d\theta &= b_n . \end{aligned}\right\} (*)$$

(*) may be regarded as the definition for <u>Fourier coefficients of f</u>. Together with (a), this gives a <u>formula for $u|_{D_1}$ as a series</u>. What about f?

This is a nontrivial question, since "the series converges to u on D_1" and "u is continuous on $\overline{D_1}$" do NOT imply that the series converges to u on ∂D_1.

Since f is (uniformly) continuous, $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $|\theta_2 - \theta_1| < \delta \Rightarrow |f(\theta_2) - f(\theta_1)| < \frac{\epsilon}{2}$. This means that for $n > \frac{2\pi}{\delta}$ (i.e. sufficiently large),

$$\left| \frac{1}{\pi} \int_{\frac{2\pi}{n} m}^{\frac{2\pi}{n}(m+1)} \underbrace{f(\theta)}_{\left(\substack{f\left(\frac{2\pi}{n} m\right) + \tilde{f}_m(\theta), \text{ where} \\ |\tilde{f}_m(\theta)| < \frac{\epsilon}{2} \text{ since } \frac{2\pi}{n} < \delta} \right)} \cos(n\theta)\, d\theta \right| = \left| \frac{1}{\pi} \int_{\frac{2\pi}{n} m}^{\frac{2\pi}{n}(m+1)} \tilde{f}_m(\theta) \cos(n\theta)\, d\theta \right| < \frac{1}{\pi} \cdot \frac{2\pi}{n} \cdot \frac{\epsilon}{2} = \frac{\epsilon}{n}$$

$$\Longrightarrow \ |a_n| = \left| \sum_{m=0}^{n-1} \frac{1}{\pi} \int_{\frac{2\pi}{n} m}^{\frac{2\pi}{n}(m+1)} f(\theta) \cos(n\theta)\, d\theta \right| < \epsilon . \qquad (**)$$

So $a_n \to 0$ as $n \to \infty$. (Similar for b_n)

But this isn't good enough for convergence, and indeed if $\mathcal{S} \subset [0, 2\pi]$ is <u>any</u> set of measure zero, then $\exists\, f \in C^0([0, 2\pi])$ whose Fourier series diverge (unboundedly!) on $\mathcal{S}$. A famous theorem of <u>Carleson</u> implies that the Fourier series at least converges pointwise almost everywhere, but still this is a bit shocking

(A) when you first learn Fourier series from physicists who repeat the mantra that $f \in C^k \Rightarrow a_n \sim \frac{1}{n^{k+2}}$

(B) in light of our theorem on Dirichlet for D_1.

The problem is that, while <u>u limits to $f(e^{i\theta})$ at each point $e^{i\theta_0} \in \partial D_1$</u>, this statement amounts to Abel summability of the Fourier series at θ_0, which is weaker than ordinary summability!

To fix this, suppose now that f is everywhere differentiable, with bounded derivative (weaker than C^1).

Then $\|f'\|_{[0,2\pi]} \le M$, and so if $\epsilon = \frac{2\pi M}{n}$ then we can take $\delta = \frac{\epsilon}{M} = \frac{2\pi}{n}$ $\underset{(**)}{\Longrightarrow}$ $|n a_n| \le \frac{2\pi M}{\epsilon} \cdot \epsilon = 2\pi M$ $(\forall n)$.

(& same for b_n).

If f is C^1, then

$\int$ by parts

$$\underset{[b_n]}{|a_n|} = \left| \frac{1}{n} \int_0^{2\pi} f(\theta) \underset{[\sin]}{\cos(n\theta)}\, d\theta \right| = \left| \underset{[+]}{-} \frac{1}{\pi n} \int_0^{2\pi} f'(\theta) \underset{[\cos]}{\sin(n\theta)}\, d\theta \right|$$

$$\le \frac{\epsilon}{n}$$

[same technique (appl. to f') as in the derivation of (**).]

and we conclude that $|n \underset{[b_n]}{a_n}| \to 0$.

In the first case ($n a_n$ bounded) we can use Littlewood's theorem, in the second case ($n a_n \to 0$) Tauber's theorem, to assert that $(\forall \theta_0)$

see Appendix

$$f(\theta_0) = \lim_{r \to 1^-} u(re^{i\theta_0})$$

$$= \lim_{r \to 1^-} \left(\frac{a_0}{2} + \sum_{n \geq 1} \left[a_n \cos(n\theta_0) + b_n \sin(n\theta_0) \right] r^n \right)$$

$$= \frac{a_0}{2} + \sum_{n \geq 1} \left[a_n \cos(n\theta_0) + b_n \sin(n\theta_0) \right]$$

(and that this last expression converges).

see Appendix

Conversely, by Abel's theorem, whenever the last expression converges, the $\lim_{r \to 1^-}$ must equal it; and since the $\lim_{r \to 1^-}$ gives $u(re^{i\theta_0}) = f(\theta_0)$, we have the

Theorem Let $f \in C^0_{\mathbb{R}}([0, 2\pi])$ and a_n, b_n be its Fourier coefficients. Then:

(i) Whenever $\frac{a_0}{2} + \sum_{n \geq 1} a_n \cos(n\theta) + b_n \sin(n\theta)$ (the Fourier series of f) converges, it converges to $f(\theta)$.

(ii) The Fourier series converges everywhere if f is everywhere differentiable, with bounded derivative.

Appendix to §I: Abel, Tauber, & Littlewood

To complete our discussion of these results, we recall what Abel & Tauber say for a function

$$f(x) := \sum a_n x^n \quad \text{on} \quad (-1,1).$$

Abel: $\sum a_n$ converges $\Rightarrow$ f has continuous extension to $(-1,1]$ (by setting $f(1) := \sum a_n$)

Tauber: f has continuous extension to $(-1,1]$ (and $na_n \to 0$) $\Rightarrow$ $\sum a_n$ converges (to $f(1)$).

So Tauber's theorem is a conditional converse.

There's a stronger version due to Littlewood, relaxing "$na_n \to 0$" to $|na_n| \leq B$.

II. Harnack's principle

As an application of the Poisson formula, we get bounds on values of a harmonic function $u \in \mathcal{H}(\overline{D}_1)$.

For instance, if $u \geq 0$ and $u(0) = 1$, then $u(\frac{3}{4}) \in [\frac{1}{7}, 7]$.

This is a special case of

$\boxed{\text{Harnack's inequality (1887)}}$ Let $u \in \mathcal{H}(\overline{D}_R)$ be nonnegative, $z \in D_R$. Then

$$\frac{R-|z|}{R+|z|} u(0) \leq u(z) \leq \frac{R+|z|}{R-|z|} u(0).$$

Remark// If the disk isn't centered at the origin, an obvious corollary (just by shifting everything) is

$$\frac{R-|z-z_0|}{R+|z-z_0|} u(z_0) \leq u(z) \leq \frac{R+|z-z_0|}{R-|z-z_0|} u(z_0).$$ //

Proof: The Poisson formula says

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \, d\theta .$$

We have

$$\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2} \le \frac{R^2-|z|^2}{(R-|z|)^2} = \frac{R+|z|}{R-|z|}$$

and

$$\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2} \ge \frac{R^2-|z|^2}{(R+|z|)^2} = \frac{R-|z|}{R+|z|}.$$

Since $u(Re^{i\theta}) \ge 0$, we can multiply both of these inequalities by this. So

$$\frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta}) \overbrace{\frac{R-|z|}{R+|z|}}^{\text{const.}}\, d\theta \le u(z) \le \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta}) \overbrace{\frac{R+|z|}{R-|z|}}^{\text{const.}}\, d\theta$$

$$\xleftarrow{\quad \text{MVT} \quad}\quad\xrightarrow{}$$

$$\frac{R-|z|}{R+|z|}u(0) \qquad\qquad \frac{R+|z|}{R-|z|}u(0)$$

$\square$

$\boxed{\text{Harnack's Principle (1887)}}$ Let U be a region, and $\{u_j\} \subset \mathcal{H}(U)$ a sequence with $u_1 \le u_2 \le \cdots$. Then $u_j \to \infty$ uniformly on compact sets (of U) OR $\exists\, u \in \mathcal{H}(U)$ s.t. $u_j \to u$ uniformly on compact sets.

Remark// So, for example, an increasing sequence of harmonic functions with $\{u_j(z_0)\}$ bounded for one $z_0 \in U$, converges to a harmonic function! This seems so surprising that when Harnack told it to Felix Klein, the latter refused to accept its validity! //

Proof: Set $U^{fin} := \{ z \in U \mid \lim u_j(z) < \infty \}$

$$U^{\infty} := \{ z \in U \mid \lim u_j(z) = \infty \}.$$

First suppose $U^{\infty} \neq \emptyset$: for $p \in U^{\infty}$, $\exists\, J$ s.t. $u_j(p) > 0$ for $j \geq J$. Clearly $\exists\, R$ s.t. $\overline{D}(p,R) \subset U$ and $u_J|_{\overline{D}}$ (hence every $u_j|_{\overline{D}}$, $j \geq J$) is positive.

So Harnack's inequality applies, and for $z \in D(p, R/2)$

$$u_j(z) \geq \frac{R - |z-p|}{R + |z-p|}\, u_j(p) \geq \frac{R - R/2}{R + R/2}\, u_j(p) = \frac{u_j(p)}{3} \longrightarrow \infty$$

and $u_j(z)$ goes uniformly to ∞ on $D(p, R/2)$.

Next suppose $\exists\, q \in U$ s.t. $u_j(q) \to \ell < \infty$, i.e. $q \in U^{fin}$ $(\neq \emptyset)$, and let $\overline{D}(q,s) \subset U$. Harnack's inequality applies to the differences, which are nonnegative, so for $z \in D(q, \frac{s}{2})$

$$0 \leq u_{j+k}(z) - u_j(z) \leq \frac{s + |z-q|}{s - |z-q|} \cdot (u_{j+k} - u_j)(q) \leq \underbrace{\cancel{\frac{s + s/2}{s - s/2}}}_{3} \underbrace{(u_{j+k}(q) - u_j(q))}_{\downarrow\, j \to \infty \;\; 0}$$

$(j, k \in \mathbb{N})$

$\Longrightarrow \{u_j\}$ uniformly Cauchy in $\|\cdot\|_{D(q, s/2)}$

$\Longrightarrow \{u_j\}$ converges pointwise to some function u (uniformly in $D(q, s/2)$)

$\Longrightarrow u$ is harmonic on $D(q, s/2)$.

(Corollary to MVT)

Conclude that U^{fin}, U^{∞} are both open.

Moreover, clearly $U = U^{fin} \amalg U^{\infty}$, and so U connected $\Longrightarrow$ U^{fin} or U^{∞} is empty.

Further, for $K \subset U$ compact, K is covered by a finite collection of balls $D(p, R/2)$ or $D(q, S/2)$ as above; and by uniformity of $u_j \to \infty$ or u on these balls, we get uniform convergence on K. $\square$