Problem Set 1

(1) Let $\{f_\alpha\}$ be a normal family of holomorphic functions on a region U. Show that $\{f'_\alpha\}$ is a normal family.

(2) Let $\beta \in D_1$ and $f(z) = \frac{z-\beta}{1-\bar{\beta}z}$. Prove that the sequence $\{f_n\}$ defined by $f_1 = f$, $f_{n+1} = f \circ f_n$ converges normally, and find the limit function. [Hint: use (1); OK to do for a subsequence.]

(3) Let $\Omega \subsetneq \mathbb{C}$ be as in the Riemann mapping theorem, with the additional assumption that Ω be symmetric with respect to the real axis. Let $f\colon \Omega \to D_1$ be a conformal isomorphism sending $p \in \Omega$ to 0, with $p \in \mathbb{R}$ and $f'(p) \in \mathbb{R}_+$. Prove that $\overline{f(\bar{z})} = f(z)$. [Hint: use the uniqueness part of the RMT.]

(4) Let $U \subset \mathbb{C}$ be a bounded region, $\{f_j\} \subset Hol(U)$ a sequence with $\int_U |f_j(z)|^2 dx\, dy < C < \infty$ (where C is independent of j). Prove that $\{f_j\}$ is a normal family. [Hint: for $z_0 \in \bar{D}(z_0, \epsilon) \subset U$, use the Cauchy integral formula to bound $|f_j(z_0)|^2$ by $C/\pi\epsilon^2$; then show the $\{f_j\}$ are locally uniformly bounded and use Lecture 1.]

(5) Let U be a region and $\mathfrak{F} = \{f_\alpha\} \subset Hol(U)$ a family with $Re(f_\alpha) > 0$ on U. Prove that $\mathfrak{F}$ is "normal in the classical sense": any sequence $\{f_n\} \subset \mathfrak{F}$ contains a subsequence converging uniformly on compact sets OR tending uniformly to ∞ on compact sets. [Hint: consider e^{-f}, and use Hurwitz.]