Problem set 11

Exercises (3)-(4) prove the theorem of Nevanlinna mentioned at the end of Lecture 33.

(1) Prove that the universal cover of $\mathbb{C}^*\backslash[1,\infty)$ cannot be $\mathbb{C}$ or $\mathbb{P}^1$.

(2) If Γ is the fundamental group (the group of homotopy classes of loops) of a Riemann surface M having $\mathfrak{H}$ as universal covering surface, then $M \cong \mathfrak{H}/\Gamma$. Prove that 2 Riemann surfaces $M_i \cong \mathfrak{H}/\Gamma_i$ $(i = 1,2)$ are isomorphic if and only if Γ_1 and Γ_2 are conjugate in $\mathrm{PGL}(2,\mathbb{R})$.

(3) Let $\mathcal{S}^* := \{f \in \mathcal{S} \mid f(D) \text{ is starlike}\}$. (A set E is starlike if for every $z \in E$, $tz \in E$ for every $t \in [0,1]$.) Prove that $Re(z\frac{f'}{f}) \geq 0$ for an $f \in \mathcal{S}^*$ by following the steps:
(a) Show $\frac{\partial}{\partial\theta}\arg(f(re^{i\theta})) > 0$, for any $r \in (0,1)$. [Hint: first show that $f(\overline{D}_r)$ is starlike, making use of the function $f^{-1}(rf(z))$ and Schwarz's Lemma.]
(b) Show that $\frac{\partial}{\partial\theta}\arg f = Re(z\frac{f'}{f})$, for $|z| = r \in (0,1)$.

(4) Again let $f \in \mathcal{S}^*$, and define $g(z) := \int_0^z f(w)\frac{dw}{w} \in \mathrm{Hol}(D)$.
(a) Show that $Re(1 + z\frac{g''}{g'}) \geq 0$. [Hint: use problem 3.]
(b) Prove $g(D)$ is convex. [Hint: show $\frac{\partial}{\partial\theta}\arg(\frac{\partial}{\partial\theta}g(re^{i\theta})) \geq 0$, and interpret this geometrically.]
(c) Writing $g(z) = z + \sum_{n\geq 2} b_n z^n$, show that each $|b_m| \leq 1$. [Hint: define $G(z) := \frac{1}{m}\sum_{j=1}^m g(\zeta_m^j z)$, where $\zeta_m = e^{2\pi i/m}$. Explain why $G(D) \subset g(D)$, then consider $h(z) := g^{-1}(G(z))$ and compare the lowest order terms in the expansions of $g(h(z))$ and $G(z)$.]
(d) Writing $f(z) := z + \sum_{n\geq 2} a_n z^n$, deduce that $|a_m| \leq m$ for each m.