# Problem Set 3 (Solutions)

(1) Show that $\frac{R^2-|z|^2}{|Re^{i\theta_0}-z|^2}$ is harmonic (in $z$) on $D_R$. [Hint: what function is it the real part of?]

(2) (i) Solve the Dirichlet problem on $\overline{D}_1$ for the following functions on $\partial D_1$. (Think of $\theta \in (-\pi, \pi]$.)

(a) $f(\theta) = \begin{cases} 1, & |\theta| \leq \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < |\theta| \leq \pi \end{cases}$,

(b) $f(\theta) = 1 - \frac{2}{\pi}|\theta|$,

(c) $f(\theta) = (|\theta| - \pi)^2 - \frac{\pi^2}{3}$.

First express your answer as a series, then try to identify the holomorphic function of which $u$ is the real part. [Hint: use part I of Lecture 7, on Fourier series.]

(ii) Using your answer to part (i), evaluate

(a) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

(b) $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots$

(c) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

(3) True or False: If $f$ is subharmonic on a region $\Omega$, then so is $|f|$.

(4) Let $\Omega \subseteq \mathbb{C}$ be open, and $\{f_j\} \subset \underline{\mathcal{H}}(U)$ a sequence of subharmonic functions comverging normally to a function $f \colon U \to \mathbb{R}$. Is $f$ necessarily subharmonic?

(5) Let $F \colon U \to V$ be holomorphic and 1-to-1, and $f \colon V \to \mathbb{R}$ be subharmonic. (Here $U$ and $V$ are regions.) Prove that $f \circ F$ is subharmonic.

(6) Let $\Omega \subseteq \mathbb{C}$ be a region, and $K \subset \Omega$ a compact subset. Show that there exists a constant $M$ (depending only on $\Omega$ and $K$) such that for every positive harmonic function $u \in \mathcal{H}(\Omega)$ and pair of points $z_1, z_2 \in K$, we have $u(z_2) \leq M \cdot u(z_1)$. [Hint: Harnack.]