Problem Set 4

(1) Show $|z|^\alpha$ is subharmonic (on $\mathbb{C}$) for every positive real α.

(2) Let $U = D_1^*$ and f be a continuous function on the boundary (unit circle together with the origin). Then the first 2 steps of Perron's proof go through: that is, if u (function on D_1^*) is the pointwise supremum of the functions in $\mathcal{S}$, then u is bounded above and harmonic. Determine u in both of the following cases:

(a) $f \equiv 1$ on the unit circle and $f(0) = 0$;

(b) $f \equiv 0$ on the unit circle and $f(0) = 1$.

If u is not in $\mathcal{S}$, exhibit a sequence of functions converging to it (which is guaranteed by the second part of the proof). [Hint: use (1), though you only need that $|z|^\alpha$ is subharmonic on D_1^*.]

(3) Let U be a bounded and simply connected region which has a barrier at each point, and let ϕ be a positive (real valued) continuous function on the boundary ∂U. Prove that there exists a holomorphic function f on U whose modulus $|f|$ extends to a continuous function on $\overline{U}$ with $|f| = \phi$ on ∂U.

(4) Carry out the following outline of proof that a bounded region U with boundary consisting of two C^1 Jordan curves (the domain being homeomorphic to an annulus) is biholomorphic to a region of the form $\{z \mid 1 < |z| < R\}$.

(a) Prove that there exists $u \in \mathcal{H}(U)$ such that the limit of u at the outer (resp. inner) boundary is 1 (resp. 0).

(b) Choose a fixed loop γ going once around the hole counterclockwise. Set $\omega = -u_y dx + u_x dy$. Show $\int_\gamma \omega \neq 0$ by supposing otherwise (i.e. $\int_\gamma \omega = 0$) and applying the argument principle to $u + \mathbf{i} \int \omega$.

(c) Pick $\lambda \in \mathbb{R}$ so that $\int_\gamma \lambda\omega = 2\pi$. Show that $\exp\{\lambda u + \mathbf{i} \int \lambda\omega\}$ is a well-defined, holomorphic, 1-to-1 and onto mapping from U to $\{z \mid 1 < |z| < e^\lambda\}$.