Problem Set 5

(1) Prove that there does not exist a Green's function for the region $\mathbb{C}$. (You may take the point z_0 to be 0.)

(2) (a) Find the Green's function $g(z, z_0)$ for the upper half-plane $\mathfrak{H}$ (with singularity at $z_o \in \mathfrak{H}$), and show it extends to a neighborhood of $\mathfrak{H} \cup \mathbb{R}$. (b) Given a continuous, real-valued function f on $\mathbb{R}$ with $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) < \infty$, show that[1] $u(z_0) := \int_{\mathcal{R}} \frac{\partial g}{\partial y}(x, z_0) f(x) dx$ solves the Dirichlet problem on $\mathfrak{H}$ with f as boundary data.

(3) Show that the period matrix for multiply connected regions (defined in my notes, or in Ahlfors) is symmetric.

(4) Let Ω be the doubly connected region obtained by removing the vertical strips $[ia, ib]$ and $[\mu + ic, \mu + id]$ from $\mathbb{C}$. This is conformally equivalent to an annulus $A(1, \lambda)$ for a unique $\lambda > 1$. (a) Determine λ. (b) Can you relate this to a complex torus $\mathbb{C}/\Lambda$? (Actually there is more than one way to do this.)

[1]The meaning of $\frac{\partial g}{\partial y}(x, z_0)$ is that we are taking the vertical partial derivative in the first variable z and evaluating at $z = x + i0$. We are now thinking of z_0 as a *variable* point in $\mathfrak{H}$.