Problem Set 6

(1) Determine $\Gamma(\frac{1}{6})$ in terms of $\Gamma(\frac{1}{3})$.

(2) Define the *digamma* function $\Psi(z) := \Gamma'(z)/\Gamma(z)$.

(a) Check that $\Psi(1+z) = \Psi(z) + \frac{1}{z}$.

(b) The Weierstrass product formula for $\Gamma(z)$ yields a summation formula for $\log\Gamma(z)$. By differentiating this formula, and using $\frac{1}{1+x} - 1 = -\sum_{k\geq 2}(-1)^k x^{k-1}$, derive the power series expansion $\Psi(1+z) = -\gamma + \sum_{k\geq 2}(-1)^k \zeta(k) z^{k-1}$.

(c) Use this to obtain a power series for $\log\Gamma(1+z)$. By evaluating this power series at $z = 1$, obtain a formula for the Euler constant γ in terms of the zeta values $\zeta(k)$.

(3) (a) Compute $\Gamma(\frac{1}{2} - n)$ for $n \in \mathbb{Z}_{>0}$.

(b) Show that $\lim_{n\to\infty} \Gamma(\frac{1}{2} - n + it) = 0$ uniformly for $t \in \mathbb{R}$.

(4) Show that for $x > 0$, $e^{-x} = \frac{1}{2\pi i}\int_{\sigma=\sigma_0} x^{-s}\Gamma(s)ds$ where $s = \sigma + it$, and the integral is taken on a vertical line with fixed real part $\sigma_0 > 0$, and $-\infty < t < \infty$. [Hint: what is the residue of $x^{-s}\Gamma(s)$ at $s = -n$? Also, use problem (3) and the Stirling formula.]

(5) Finish the proof that $\xi(s) = \xi(1-s)$ from Thm. 2 of Lecture 17.