Problem Set 7

(1) Show that we cannot have $\liminf_{x\to\infty} \frac{\varphi(x)}{x} < 1$. (See Theorem 3 of Lecture 19 for context and hint.)

(2) Prove that for $r > s$ non-negative integers, $\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} dx\, dy \in \frac{1}{d_r^2}\mathbb{Z}$ while $\int_0^1 \int_0^1 \frac{x^r y^r}{1-xy} dx\, dy = \zeta(2) - \frac{1}{1^2} - \cdots - \frac{1}{r^2}$. [Hint: use the proof of Lemma 1 in Lecture 20. This is easy.]

(3) Prove $\zeta(2)$ (and hence π, since $\zeta(2) = \frac{\pi^2}{6}$) is irrational, by following the outline:
(a) Show that $\int_0^1 \int_0^1 \frac{(1-y)^n P_n(x)}{1-xy} dx\, dy = (-1)^n \int_0^1 \int_0^1 \frac{\phi(x,y)}{1-xy} dx\, dy$, where $\phi(x,y) = \frac{xy(1-x)(1-y)}{1-xy}$.
(b) Show that on $[0,1]^2$, ϕ is bounded by $(\frac{\sqrt{5}-1}{2})^5$.
(c) Using (2), parts (a) and (b), and Lemma 2 of Lecture 20, derive a contradiction from $\zeta(2) = \frac{P}{Q}$ ($P, Q \in \mathbb{Z}$, $Q \neq 0$).

(4) Given a Dirichlet series $\sum_{n\geq 1} \frac{a_n}{n^s}$ (here $\{a_n\} \subset \mathbb{C}$ is just some sequence) converging for some $s_0 = \sigma_0 + it_0$, it converges normally on the set $\{s \mid Re(s) > \sigma_0\}$.

(5) Let $F(s) = \sum_p p^{-s}$, where the sum runs over all primes. Show that $F(s) = \log\zeta(s) + G(s)$, where log denotes the principal branch of the logarithm, and $G(s)$ is analytic on the half-plane $Re(s) > \frac{1}{2}$. Deduce from this that the function $F(s)$ does *not* have a meromorphic continuation to the left of the line $\sigma = 1$.

(6) Show that, if x is sufficiently large, then the interval $[2, x]$ contains more primes than the interval $(x, 2x]$.