Problem Set 8

Let ω_1, ω_2 be two complex numbers, linearly independent over $\mathbb{R}$, and $\Lambda = \mathbb{Z}\langle \omega_1, \omega_2 \rangle \subset \mathbb{C}$ be the lattice they generate.

(1) Show that the series

$$\frac{1}{z^2} + \sum\nolimits'_{\omega\in\Lambda} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

defining $\wp(z)$ is absolutely and uniformly convergent on any compact subset of $\mathbb{C}$ which does not contain any of the points of Λ. Try to do this without the Lemma at the end of Lecture 22. (Perform a direct estimate.)

(2) There are three perspectives on the addition theorem for $\wp$: analytic, geometric, and algebraic. I'll do the latter two in class and you'll provide the analytic approach here, by following the steps below. Consult Ahlfors pp. 276-77 for hints.

(a) $\wp(z) - \wp(u) = -\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2}$

(b) $\frac{\wp'(z)}{\wp(z)-\wp(u)} = \zeta(z-u) + \zeta(z+u) - 2\zeta(z)$

(c) $\zeta(z+u) = \zeta(z) + \zeta(u) + \frac{1}{2}\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}$

(d) $\wp(z+u) = -\wp(z) - \wp(u) + \frac{1}{4}\left(\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}\right)^2$ [addition formula]

(e) $\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z)$

(3) For this problem, assume that Λ is such that $g_2 = -4$ and $g_3 = 0$. (That is, $\mathbb{C}/\Lambda$ is isomorphic to the algebraic curve $y^2 = 4x^3 + 4x$. Such a Λ exists simply by taking it to be the set of all periods of $\frac{dx}{y}$ on this curve.)

(a) Express the RHS of formula 2(e) as a rational function of $\wp(z)$.

(b) Let u be such that $\wp(u) = 1$ (and $\wp'(u) = 2\sqrt{2}$). Show that u is 4-torsion, i.e. $4u \in \Lambda$.

(c) Let u be such that $\wp(u) = \frac{p}{2^a q}$, where the fraction is written in lowest terms, a is an odd natural number, and p, q are odd integers. Show that u is of "infinite order", i.e. no integer multiple of it lies in Λ. [Hint: put $u_0 := u$. Show that $\wp(u_1)$, where $u_1 = 2u_0$, is of the same form, with bigger a, and iterate. Then suppose u_0 was N-torsion and produce a contradiction via the pigeonhole principle.]

(4) This exercise concerns the Theta function, defined in Lecture 22. (See also Lecture 23 for η_1.)

(a) Check that $\theta(z+1) = \theta(z)$.

(b) Check that $\theta(z) = Ce^{h(z)}\sigma(z + \frac{\tau+1}{2})$, where $h(z) = -\frac{\eta_1}{2}z^2 - (\frac{\eta_1\tau}{2} + \frac{\eta_1}{2} + \pi i)z$.