Problem Set 9

(1) Given any $p, q \in \{0, 1, \ldots, N-1\}$ (not both 0), show that

$$f(\tau) := \sum_{(m,n) \underset{(N)}{\equiv} (p,q)} \frac{1}{(m\tau + n)^k}$$

belongs to $M_k(\Gamma(N))$.

(2) Consider the Legendre elliptic curve $Y^2 = X(X-1)(X-\lambda)$. This *Legendre form* contains slightly more information than the isomorphism class (of elliptic curve), since the 2-torsion points are essentially ordered (or "marked"): $(0,0)$, $(1,0)$, and $(\lambda, 0)$. Passing to the isomorphism class of the elliptic curve *without* marked 2-torsion, by taking the j-invariant, forgets the ordering of these 3 points. So $\lambda \mapsto j(\lambda)$ should be a 6-to-1 map, since $\mathcal{S}_3$ has order 6. In this problem you'll find that map explicitly, thereby obtaining a conceptual derivation of the $\phi(\lambda)$ from Lecture 27.

(a) By the affine change of coordinates $Y = y/2$, $X = x + \frac{\lambda+1}{3}$, put the Legendre curve in Weierstrass form $y^2 = 4x^3 - g_2 x - g_3$. In so doing, you get g_2 and g_3 as explicit functions of λ.

(b) These functions aren't really well-defined: there are further changes of coordinates that will transform $(g_2, g_3) \mapsto (\xi^4 g_2, \xi^6 g_3)$, as you can check. But we know the j-invariant $\frac{g_2^3}{g_2^3 - 27g_3^2}$ *is* well-defined. Compute it as a function of λ using your result from (a); this should coincide with $\phi(\lambda)$.

(3) Show that the λ-function, which we defined on $\mathfrak{H}$, actually does not. analytically continue along any path meeting the real axis. (Hence $\mathfrak{H}$ is truly its "natural domain".) To do this, use part of the idea of the proof of little Picard.

(4) Consider the polylogarithm functions

$$Li_n(z) := \sum_{k \geq 1} \frac{z^k}{k^n},$$

which are defined *a priori* in D_1. (Of course $Li_1(z) = -\log(1-z)$.)

(a) Write Li_2 as an integral and use this to continue it to a holomorphic function on $\mathbb{C} \setminus [1, \infty)$. (You need to use the monodromy theorem here.) Iterate the procedure for the other Li_n, $n > 2$.

(b) What is the "monodromy" of Li_n about $z = 1$, i.e. what analytic function in (part of) the disk does it yield when continued around this point once counterclockwise? (Use the integral expressions from part (a).)