I. Sets

I.A. Relations

Recall that a **set** S is a collection of *elements*. If its **order** $|S|$ (i.e., the number of elements) is finite, we may list its elements as in $S = \{s_1, \ldots, s_n\}$; alternatively, we may write $s_1 \in S$ to say "s_1 is an element of S". A collection T of *some* elements of S is called a *subset*, written $T \subset S$. A *proper* subset $T \subsetneq S$ is one which is not S itself. The empty set $\emptyset$ contains no elements and is a subset of every set.

If S is the union of subsets $\{S_i\}_{i \in I}$,[1] we will write $S = \bigcup_{i \in I} S_i$. When these sets are disjoint (viz., $S_i \cap S_j = \emptyset$ for $i \neq j$), writing instead $S = \coprod_{i \in I} S_i$ conveys that information. If the S_i are also nonempty, then this defines a **partition** of S.

Let S, T be sets. A **map** (or *mapping*, or *function*)

$$f \colon S \to T$$

is a rule associating to each $s \in S$ an element $f(s) \in T$; it has *graph*

$$\Gamma_f = \{(s, f(s)) \mid s \in S\}.$$

A subset $\Gamma \subset S \times T$ is the graph of some map if and only if

$$\begin{cases} \forall s \in S \ \exists t \in T \text{ such that } (s,t) \in \Gamma, \\ \quad \text{and} \\ (s,t), (s,t') \in \Gamma \implies t = t'. \end{cases}$$

We say that:

- f is **injective** (written $f \colon S \hookrightarrow T$) if $f(s) = f(s') \implies s = s'$;
- f is **surjective** (written $f \colon S \twoheadrightarrow T$) if $f(S) = T$; and

[1] Here I is called an *index set*; here, to each element of I there is associated a subset $S_i \subset S$.

- f is **bijective** (written $f\colon S \xrightarrow{\cong} T$) if f is injective and surjective (and we define its *inverse map* $f^{-1}\colon T \xrightarrow{\cong} S$ to send each $f(s) \mapsto t$).

Composition of maps

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[rr, bend right, "g\circ f \text{ (or } gf)"'] & T \arrow[r, "g"] & U \end{tikzcd}$$

is inherently associative. When inverses exist, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

I.A.1. DEFINITION. (i) A **relation** on S is a subset

$$\sim \subset S \times S.$$

If $(a,b) \in \sim$, then we write "$a \sim b$".

(ii) $\sim$ is an **equivalence relation** if

$$\begin{cases} \text{(reflexivity)} \;\; a \sim a \\ \text{(symmetry)} \;\; a \sim b \implies b \sim a \\ \text{(transitivity)} \;\; a \sim b \text{ and } b \sim c \implies a \sim c \end{cases}$$

hold for all $a, b, c \in S$.

I.A.2. EXAMPLES. Here are some random equivalence relations.

(i) $\mathcal{P}$ = set of all people; $p_1 \sim p_2 \iff p_1, p_2$ reside in same country.[2]

(ii) "$=$" on any set S.

(iii) On $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, $p \sim q \iff p, q$ are equidistant from $(0,0)$; or $p \equiv q \iff p, q$ lie on the same horizontal line.

(iv) Write $\mathbb{N} = \{0, 1, 2, \ldots\}$ for the natural numbers. On $\mathbb{N}^2$, say $(a,b) \sim (c,d) \iff a + d = b + c$.

(v) On the integers $\mathbb{Z}$, define "$\underset{(n)}{\equiv}$" (or "$\equiv$ (mod n)") by

$$a \underset{(n)}{\equiv} b \iff n \mid a-b.$$

(vi) Given $f\colon S \to T$, define (on S)

$$a \sim_f b \iff f(a) = f(b).$$

[2]Note: "$\iff$" means "iff" (i.e., if and only if). We use it here to define things.

I.A.3. NON-EXAMPLES. Here are some relations which are *not* equivalence relations.
(i) On $\mathcal{P}$, $p_1 \sim p_2 \iff p_1, p_2$ are cousins.
(ii) On $\mathbb{R}, \mathbb{N}, \mathbb{Q}$: $>, \geq$.
(iii) On $\mathbb{Z}$, $a \sim b \iff a$ relatively prime to b.
(iv) On $\mathbb{Z}$, $a \mid b \iff a$ divides b.

Given an equivalence relation $\sim$ on S, the **$\sim$-equivalence class** of $a \in S$ is

$$\bar{a} := \{b \in S \mid b \sim a\} \subset S. \tag{I.A.4}$$

I.A.5. PROPOSITION. *The $\sim$-equivalence classes yield a partition of S, and every partition arises in this way.*

PROOF. See Exercise (4) of Problem Set 1. □

I.A.6. DEFINITION. (i) The **quotient set**

$$S/\sim \;:=\; \{\bar{a} \mid a \in S\} \subset \mathscr{P}(S)$$

is the set of $\sim$-equivalence classes.[3]

(ii) The **natural map** $\nu\colon S \to S/\sim$ sends $a \mapsto \bar{a}$.

We shall say two sets are **isomorphic**, written $S \cong T$, if there is a bijective map between them.

I.A.7. EXAMPLES. Referring to I.A.2,
(i) $\mathcal{P}/\sim \;\cong$ set of all countries.
(ii) $S/= \;\cong S$.
(iii) $\mathbb{R}^2/\sim \;\cong \{$circles with center $(0,0)\} \cong \mathbb{R}_{\geq 0}$;
and $\mathbb{R}^2/\equiv \;\cong \{$horizontal lines$\} \cong \mathbb{R}$.
(iv) $\mathbb{N}^2/\sim \;\cong \mathbb{Z}$.
(v) $\mathbb{Z}/\underset{(n)}{\equiv} \cong \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1}\}$ is the set of **residue classes mod n**.

Finally, given sets S, T, an equivalence relation $\sim$ on S, and a map $f\colon S \to T$, we have:

[3] Here $\mathscr{P}(S)$ denotes the set of subsets of S, called its *power set*.

I.A.8. PROPOSITION. *Suppose that*

$$a \sim b \implies f(a) = f(b)$$

for all $a, b \in S$*. Then there is a unique* $\bar{f} \colon S/{\sim} \to T$ *such that* $\bar{f} \circ \nu = f$*.*

PROOF. Define $\bar{f}$ to send $\bar{a} \in S/{\sim}$ to $f(a) \in T$. This is well-defined (i.e., doesn't depend on the choice of representative of the equivalence class), and no other choice makes the diagram commute. □

In the scenario of I.A.8, we say that f *is well-defined mod(ulo)* $\sim$, or that the diagram

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "\nu"'] & & T \\ & S/{\sim} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

commutes. As a simple example, consider the map $f \colon \mathbb{Z} \to \{1, -1\}$ sending $n \mapsto (-1)^n$, which is well-defined "mod 4", i.e. modulo $\underset{(4)}{\equiv}$. So $\bar{f} \colon \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\} \to \{-1, 1\}$ sends $\bar{0}, \bar{2} \mapsto 1$ and $\bar{1}, \bar{3} \mapsto -1$. Obviously this works for any other even integer; in particular, if we take $\sim$ to be $\underset{(2)}{\equiv}$, then $\bar{f}$ is an isomorphism.