II.C. Groups and subgroups

Some further simple properties follow from the defining properties:

II.C.1. PROPOSITION. *Let G be a group, and $a, b, x \in G$.*
(a) *The cancellation laws hold:* $xa = xb$ (*or* $ax = bx$) $\implies a = b$.
(b) *The inverse of x is unique, and* $(x^{-1})^{-1} = x$.
(c) $(a^n)^m = a^{nm}$, $a^m a^n = a^{m+n}$ [*laws of exponents*]
(d) *If a and b commute* ($ab = ba$), *then* $(ab)^n = a^n b^n$.

PROOF. (a) Multiply on the left (resp. right) by x^{-1}.
(b) If $x'x = 1 = xx'$ and $x''x = 1 = xx''$, then

$$x'' = x''1 = x''xx' = 1x' = x'.$$

(c) Clear from the definition: $a^n = a \cdots a$ (n times).
(d) If a commutes with b, it commutes with powers of b. Now induce on n: $(ab)^n = (ab)^{n-1}ab = a^{n-1}b^{n-1}ab = a^{n-1}ab^{n-1}b = a^n b^n$. $\square$

II.C.2. REMARK. (i) $ab = ba$ is equivalent to the triviality of the **commutator** $[a, b] := a^{-1}b^{-1}ab$. (In algebra, an element being *trvial* means it's the identity element.)
(ii) For monoids: (a) is false, (c) and (d) hold. For those elements of the monoid that *have* a (two-sided) inverse, (b) is true. (But those elements form a group, so this doesn't say much...)

II.C.3. EXAMPLES. (i) **Abelian groups**:

- $(\mathbb{A}, +, 0)$ where $\mathbb{A} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.
- $(V, +, \vec{0})$ where V is a vector space.
- $(\mathbb{Z}_n, +, \bar{0})$ where $\mathbb{Z}_n = \mathbb{Z}/\underset{(n)}{\equiv}$ = integers mod n.
- $(\mathbb{Z}_n^*, \bullet, \bar{1})$ where $\mathbb{Z}_n^* \subset \mathbb{Z}_n$ is the subset of elements possessing a multiplicative inverse: $\bar{b} \in \mathbb{Z}_n$ such that $\bar{a}\bar{b} (= \overline{ab}) = \bar{1}$.
- $(\mathbb{A}^*, \bullet, 1)$ where $\mathbb{A}^* = \mathbb{Q}^*, \mathbb{R}^*, \mathbb{C}^*$ (here $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ etc.).
- $(\{1, -1\}, \bullet, 1)$, and more generally $(\{e^{\frac{2\pi i k}{n}}\}_{k=0}^{n-1}, \bullet, 1)$.
- rotational symmetries of the (regular) n-gon.

Notes: (a) $\mathbb{Z}_n^* = \{\bar{a} \mid (a, n) = 1\}$, since (by I.B.4) $(a, n) = 1 \iff \exists b, k \in \mathbb{Z}$ with $ab + nk = 1 \iff \exists b$ such that $\overline{ab} = \bar{1}$.

(b) $\mathbb{Z}_n$ is an example of a **cyclic group**, i.e. a group on one *generator*: the notation

$$\mathbb{Z}_n = \langle \bar{1} \mid n \cdot \bar{1} = \bar{0} \rangle$$

means that the elements comprise all of the "powers" $\bar{0}, \bar{1}, \bar{1}+\bar{1}, \bar{1}+\bar{1}+\bar{1}$, etc. of the generator $\bar{1}$, subject to the *relation* shown ($n \cdot \bar{1} = \bar{1} + \cdots + \bar{1}$ [n times] $= \bar{0}$). $\mathbb{Z} = \langle 1 \rangle$ is also a cyclic group (with no relation), but (unlike $\mathbb{Z}_n$) an *infinite* one.

(ii) **Non-abelian groups**:

- $\mathfrak{S}_n = n^{\text{th}}$ symmetric group, for $n \geq 3$.
- $D_n = n^{\text{th}}$ **dihedral group**, for $n \geq 3$: its elements comprise the n rotational and n reflectional symmetries of a regular n-gon.
- $GL_n(\mathbb{A})$ **general linear group**, for $n \geq 2$ (and $\mathbb{A} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$): elements are invertible $n \times n$ matrices with entries in $\mathbb{A}$.
- $SL_2(\mathbb{Z})$ (integer 2×2 matrices with determinant 1) and other "arithmetic groups".

Notes: As suggested in (i), it can be useful to write groups in terms of *generators* and *relations*. For instance, for the "quotient of $SL_2(\mathbb{Z})$ by $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$",

$$PSL_2(\mathbb{Z}) = \langle S, R \mid S^2 = 1 = R^3 \rangle \text{ where } \begin{cases} S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ R = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = S \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{cases}$$

says that the elements of $PSL_2(\mathbb{Z})$ are arbitrary "words" in S and R (and their inverses) subject only to the two relations written. For the dihedral group, we have

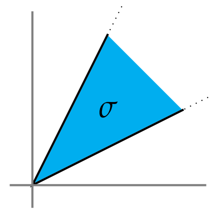
$$D_n = \langle r, h \mid \text{relations are a HW exercise!} \rangle$$

where r is counterclockwise rotation by $\frac{2\pi}{n}$ and h is a choice of reflection. We have also shown that $\mathfrak{S}_n$ is *generated* by transpositions.

(iii) **Monoids that are not groups**:

- $(\mathbb{N}, +, 0)$, $(\mathbb{Z}_{>0}, \bullet, 1)$, or $(\mathbb{Z}\backslash\{0\}, \bullet, 1)$.

- $(\mathscr{P}(S), \cup, \emptyset)$ for any nonempty set S.
- $(\sigma, +, (0,0))$ where σ is a cone in $\mathbb{R}^2$:



- the monoid of integral ideals in an algebraic number ring (which we will meet later).

(iv) **Direct products of (monoids or) groups**: $G_1 \times G_2$, with group operation $(g_1, g_2) \cdot (h_1, h_2) := (g_1 h_1, g_2 h_2)$.

II.C.4. DEFINITION. A **subgroup** of G is a subset $H \subset G$ satisfying:
(i) $1_G \in H$;
(ii) [closure under multiplication] $x, y \in H \implies xy \in H$; and
(iii) [closure under inversion] $x \in H \implies x^{-1} \in H$.
We write $H \leq G$ (or $H < G$ for a *proper* subgroup — i.e. $H \neq G$), and endow H with the operation "$\bullet$" inherited from G (and hence with a group structure).

II.C.5. EXAMPLES. (a) When $\alpha \in G$ is an element of a group, we will use the notation $\langle \alpha \rangle := \{\alpha^n \mid n \in \mathbb{Z}\}$ to denote the **cyclic subgroup** generated by α. (Though no relation is written, this can certainly be finite since some power of α may be 1 in G.) Cyclic subgroups are clearly abelian.

(b) In D_n, we have cyclic subgroups $\langle r \rangle < D_n$ (resp. $\langle h \rangle$) of order n (resp. 2). In $\mathbb{C}^*$, $\langle e^{\frac{2\pi \mathbf{i}}{n}} \rangle$ is the (cyclic) group of n^{th} roots of unity. We can intuitively think of $\langle e^{\frac{2\pi \mathbf{i}}{n}} \rangle$ and $\langle r \rangle$ as copies of $(\mathbb{Z}_n, +, \bar{0})$ embedded in $\mathbb{C}^*$ and D_n, but we'll need to employ homomorphisms and isomorphisms to state this properly.)

(c) Intersections of subgroups are again subgroups: given $H, K \leq G$, we have $H \cap K \leq G$. (Why?)

(d) Generalizing (a), we can consider subgroups generated by a *subset* $S \subset G$, denoted $\langle S \rangle \leq G$. There are three equivalent definitions of this: as the smallest subgroup of G containing S; as the intersection of all subgroups containing S; or as all products of (powers of) elements of S and their inverses.

(e) The **centralizer** of a subset $S \subset G$ is defined by

$$C_G(S) := \{g \in G \mid gs = sg \ (\forall s \in S)\} \leq G.$$

(To see that it is a subgroup, rewrite the condition in the braces as $sgs^{-1} = g$. If also $sg's^{-1} = g'$, then $s(gg')s^{-1} = (sgs^{-1})(sg's^{-1}) = gg'$, and $sg^{-1}s^{-1} = (sgs^{-1})^{-1} = g^{-1}$.) In particular, we write $C_G(a) := C_G(\{a\})$ for the centralizer of one element, and $C(G) := C_G(G)$ for the **center** of G. (Often "C" is written "Z" — this is the German heritage.)

(f) The cone in II.C.3(iii) is a submonoid of $\mathbb{R}^2$.

(g) A submonoid of $\mathfrak{T}_X$ is called a *monoid of transformations* of X. A subgroup of $\mathfrak{S}_X$ is a *group of permutations* of X. Here is an interesting example.

Define $\mathfrak{A}_n \subset \mathfrak{S}_n$ by

$$\mathfrak{A}_n := \{\alpha \in \mathfrak{S}_n \mid \alpha \text{ is even}\} = \{\alpha \in \mathfrak{S}_n \mid \text{sgn}(\alpha) = 1\}.$$

We claim that, since sgn is a homomorphism, this is a subgroup: indeed, $1 \in \mathfrak{A}_n$; and given $\alpha, \beta \in \mathfrak{A}_n$,

$$\text{sgn}(\alpha) = 1 = \text{sgn}(\beta) \implies \begin{cases} \text{sgn}(\alpha\beta) = \text{sgn}(\alpha)\text{sgn}(\beta) = 1 \\ \text{sgn}(\alpha^{-1}) = \text{sgn}(\alpha)^{-1} = 1 \end{cases}$$

so that (ii), (iii) in II.C.4 hold. This subgroup $\mathfrak{A}_n \leq \mathfrak{S}_n$ is called the **alternating group**.

II.C.6. PROPOSITION. *If $n \geq 3$, $\mathfrak{A}_n$ is generated by 3-cycles.*

PROOF. $\alpha \in \mathfrak{A}_n \implies \alpha$ is a product of an even number of transpositions. We can group these into pairs of distinct transpositions, viz. $\alpha = (\tau_1\tau_2)\cdots(\tau_{2q-1}\tau_{2q})$. For a pair $\tau\tau'$, if the transpositions are *not* disjoint, write

$$(ij)(ik) = (ikj);$$

while if they *are* disjoint, write

$$(ij)(k\ell) = (ij)\underbrace{(jk)(jk)}_{1}(k\ell) = (ijk)(jkl).$$

This recasts α as a product of 3-cycles. (That, conversely, all 3-cycles belong to $\mathfrak{A}_n$ is clear from the first displayed formula.) $\square$