II.F. Group actions and Cayley's theorem

II.F.1. DEFINITION. Let X be a set and G a group. An **action of G on X** is a function

$$G \times X \to X$$
$$(g, x) \mapsto g.x$$

satisfying:
(i) $(gh).x = g.(h.x)$ for all $g, h \in G$ and $x \in X$; and
(ii) $1_G.x = x$ for all $x \in X$.
A set X with G-action is called a *G-set*.

II.F.2. PROPOSITION. *A G-action on* X *is the same thing as a homomorphism* $\varphi\colon G \to \mathfrak{S}_X$.

PROOF. *Given* an action, $x \mapsto g.x$ is a permutation of X (i.e. bijection from X to itself), since

$$g^{-1}.(g.x) \overset{\text{(i)}}{=} (g^{-1}g).x = 1.x \overset{\text{(ii)}}{=} x \implies g \text{ 1-to-1}$$
$$g.(g^{-1}.x) \overset{\text{(i)}}{=} (gg^{-1}).x = 1.x \overset{\text{(ii)}}{=} x \implies g \text{ onto.}$$

Setting $\varphi(g)x := g.x$ therefore exhibits $\varphi(g)$ as an element of $\mathfrak{S}_X$. This is a homomorphism because

$$\varphi(g)\varphi(h)x = g.(h.x) \overset{\text{(i)}}{=} (gh).x = \varphi(gh)x \quad (\forall x)$$
$$\implies \quad \varphi(g)\varphi(h) = \varphi(gh).$$

Conversely, given φ, define $g.x = \varphi(g)x$. □

I find $\varphi(g)x$ more notationally confusing than $g.x$, but viewing an action as a homomorphism $\varphi\colon G \to \mathfrak{S}_X$ is conceptually useful. If φ is injective, we call the action *faithful* or *effective*. In that case the action presents G as a subgroup of $\mathfrak{S}_X$ (cf. II.E.5).

II.F.3. DEFINITION. Let G act on X. The **orbit** of x is the subset

$$G(x) := \{g.x \mid g \in G\} \subset X$$

consisting of its "G-translates", and the **stabilizer** of x is the subgroup[11]

$$G_x := \{g \in G \mid g.x = x\} \leq G$$

of elements "fixing" x. The action of G is **transitive** if $G(x) = \mathrm{X}$ (for some, hence any, x).

II.F.4. EXAMPLES. **(i)** $G = (\mathbb{Z}, +)$ acts on $\mathrm{X} = \mathbb{R}$ by translation:

$$n.r := r + n \quad (r \in \mathbb{R},\ n \in \mathbb{Z}).$$

[Check: $(n_1 + n_2).r = r + n_1 + n_2 = (n_1.r) + n_2 = n_2.(n_1.r)$; and $0.r = r + 0 = r$.] Let $x \in \mathbb{R}$. The orbit is $G(x) = \{x + n \mid n \in \mathbb{Z}\}$ and the stabilizer is $G_x = \{0\}$.

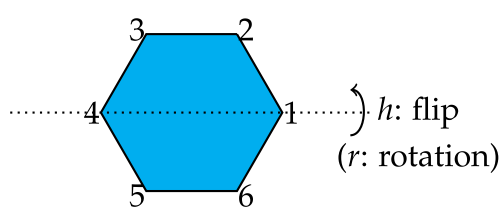
(ii) "Tautological" examples:

- $G = \mathfrak{S}_n$ acts on $\mathrm{X} = \{1, \ldots, n\}$ by

$$\sigma.a := \sigma(a) \quad (\sigma \in \mathfrak{S}_n,\ a \in \mathrm{X}).$$

  We have $G(a) = \mathrm{X}$ and $G_a \cong \mathfrak{S}_{n-1}$, where the $\mathfrak{S}_{n-1}$ arises from permutations of $\{1, \ldots, n\} \setminus \{a\}$.
- $GL_n(\mathbb{R})$ acts on $\mathrm{X} = \mathbb{R}^n$ by matrix multiplication.

(iii) D_6 acts on $\mathrm{X} = \{1, \ldots, 6\}$ by viewing X as the vertices of a regular hexagon:



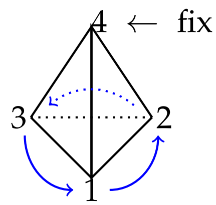
It's helpful to use homomorphism notation here:

$$\varphi(r) = (123456)$$
$$\varphi(h) = (26)(35).$$

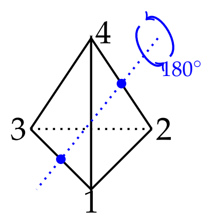
Since D_6 is generated by r, h, $\varphi(D_6) = \langle (123456), (26)(35) \rangle \leq \mathfrak{S}_6$.

[11][**Jacobson**]'s notation: $\mathrm{Stab}(x)$.

(iv) The group S_T of *rotational* symmetries of the regular tetrahedron acts faithfully on its set of vertices $X = \{1, 2, 3, 4\}$. Viewing this as an embedding (i.e. injective homomorphism) $\varphi\colon S_T \hookrightarrow \mathfrak{S}_4$ realizes S_T as $\mathfrak{A}_4$. This is because $\varphi(S_T)$ contains all the 3-cycles, like (123), which we may see as follows:



Since 3-cycles already generate $\mathfrak{A}_4$, we don't need more pictures, but the other type of element — the products of disjoint transpositions — can be visualized too, e.g. $(13)(24)$:



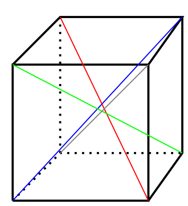
On the other hand there are no single transpositions such as (12). (We are not allowing reflections.) So

$$\mathfrak{A}_4 \leq \varphi(S_T) < \mathfrak{S}_4 \implies \mathfrak{A}_4 = \varphi(S_T),$$

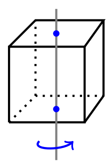
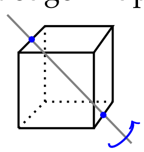
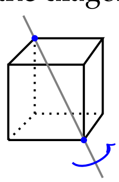
using Lagrange's theorem (how?).

(v) One can play the same game with the group S_C of rotational symmetries of the cube, acting (faithfully) on ...

- ... the vertex set: $S_C \hookrightarrow \mathfrak{S}_8$
- ... the edge set: $S_C \hookrightarrow \mathfrak{S}_{12}$
- ... the face set: $S_C \hookrightarrow \mathfrak{S}_6$
- ... the set of interior diagonals: $S_C \hookrightarrow \mathfrak{S}_4$



where "interior diagonals" connect antipodal points of the cube, as shown. Let X be the 4-element set comprising these diagonals. The following table describes the non-identity elements of S_C:

rotation type	action on X	# of possible axes	possible angles	total # of elements
about facet midpoints 	4-cycles, $(\cdot\cdot)(\cdot\cdot)$'s	3	$90°$, $180°$ $270°$	$3 \cdot 3 = 9$
about edge midpoints 	2-cycles	6	$180°$	$6 \cdot 1 = 6$
about vertices (on the diagonals) 	3-cycles	4	$120°$, $240°$	$4 \cdot 2 = 8$

Adding the identity, we see that S_C has at least 24 elements. Since the action on X is faithful, it can have at most $4! = 24$ elements. Applying II.E.7 to the homomorphism $\varphi \colon S_C \hookrightarrow \mathfrak{S}_4$, we see that $S_C \cong \mathfrak{S}_4$.

The example just concluded should have convinced you that there are *many* natural ways of looking at some groups as subgroups of permutation groups. But there is one "canonical" way:

II.F.5. CAYLEY'S THEOREM. *Every group G is a subgroup of the symmetric group $\mathfrak{S}_G$. (In particular, if $|G| = n$ is finite, then G is a subgroup of $\mathfrak{S}_n$.)*

PROOF. Let G act on itself ($X = G$) by *left translation*:

$$g.g' := gg'.$$

Clearly $(gh).g' = ghg' = g.(h.g')$ because group multiplication is associative; and also $1.g' = 1g' = g'$. By II.F.2, this yields a homomorphism $\varphi\colon G \to \mathfrak{S}_G$. It is injective because if $g \in G$ has $\varphi(g) = \mathrm{id}_G$, then $g = g1 = g.1 = \varphi(g)1 = \mathrm{id}_G(1) = 1$. So φ gives an isomorphism from G onto its image $\varphi(G) \leq \mathfrak{S}_G$. □

II.F.6. REMARK. We could also have used *right translation* in the proof, i.e.

$$g.g' := g'g^{-1}.$$

This works because

$$g.(h.g') = g.(g'h^{-1}) = g'h^{-1}g^{-1} = g'(gh)^{-1} = (gh).g'.$$

Notice that the actions in the proof and remark aren't so interesting: the orbit of any element is the entire group. Fortunately, groups also act on themselves in a more interesting way, which is our next topic.