## II.N. "Not-Burnside's" counting lemma

Indeed, Burnside himself (1897) attributed it to Frobenius (1887), though it was known much earlier to Cauchy as well. And of course, it is known to you by HW #3. We begin by reviewing the statement and proof.

II.N.1. NOTATION. Throughout, $G$ denotes a finite group acting on a finite set X, with:

- $G(x) \subset \mathrm{X}$ the **G-orbit of x** $\in \mathrm{X}$ $(= \{g.x \mid g \in G\})$;
- $\mathrm{X}/G$ := the **set of G-orbits**;
- $\mathrm{X}^g$ := the **fixed-point set of g** $\in G$ $(= \{x \in \mathrm{X} \mid g.x = x\})$; and
- $G_x \leq G$ the **stabilizer of x** $\in \mathrm{X}$ $(= \{g \in G \mid g.x = x\})$.

II.N.2. THEOREM (Burnside's Lemma). $|\mathrm{X}/G| = \frac{1}{|G|}\sum_{g\in G}|\mathrm{X}^g|$.

PROOF. Consider the subset of $G \times \mathrm{X}$

$$\mathcal{S} := \{(g,x) \mid g.x = x\} \underset{\text{set}}{\cong} \coprod_{g\in G} \mathrm{X}^g \underset{\text{set}}{\cong} \coprod_{x\in \mathrm{X}} G_x.$$

By the orbit-stabilizer theorem, $|G_x||G(x)| = |G|$; hence

$$\sum_{g\in G}|\mathrm{X}^g| = |\mathcal{S}| = \sum_{x\in\mathrm{X}}|G_x| = |G|\sum_{x\in\mathrm{X}}\frac{1}{|G(x)|}. \tag{II.N.3}$$

Taking $\{x_i\}_{i=1}^r$ to be a system of representatives of the orbits, so that $\mathrm{X} = \amalg_{i=1}^r G(x_i)$, we get (II.N.3) =

$$|G|\sum_{i=1}^r \frac{|G(x_i)|}{|G(x_i)|} = |G|r = |G||\mathrm{X}/G|.$$

□

II.N.4. REMARK. This can be extended to an infinite group $\mathcal{G}$ (acting on a finite set) by noticing that the homomorphism $\varphi\colon \mathcal{G} \to \mathfrak{S}_\mathrm{X}$ factors through $\bar{\varphi}\colon G \to \mathfrak{S}_\mathrm{X}$ where $G := \mathcal{G}/\ker(\varphi)$ is finite. One has that $|\mathrm{X}/\mathcal{G}| = |\mathrm{X}/G|$, while the RHS of Burnside is replaced by an "integral" over $\mathcal{G}$.

Though the applications we shall give are indeed about counting things, there are numerous theoretical corollaries and extensions of this result:

- In combinatorics, a refinement called *Pólya's enumeration theorem* breaks the single number $|X/G|$ out by "weights" that one attributes to elements of X. That is, there is a weight function $w\colon X \to \mathbb{N}$ (or even $\mathbb{N}^k$) constant on $G$-orbits and one wants to count orbits by weight.
- In topology, if $f\colon \mathscr{T} \to \mathscr{S}$ is a finite (connected, nonisomorphic) covering of a topological space $\mathscr{S}$, then there is a continuous map from a circle into $\mathscr{S}$ that does not lift to (i.e. factor through) $\mathscr{T}$. (Here $G$ is the "fundamental group" $\pi_1(\mathscr{S})$ and X is $f^{-1}(s)$ for some $s \in \mathscr{S}$.)
- In number theory, if a polynomial $F$ with integer coefficients (degree $\geq 2$, irreducible over $\mathbb{Q}$) has $N_p$ roots mod $p$, the density of primes for which $N_p = 0$ is at least $\frac{1}{n}$. (Here $G$ is the "Galois group" of the polynomial, and X is the set of roots of $F$ in the algebraic closure $\bar{\mathbb{Q}}$. We will meet these notions in Algebra II.)

Before proceeding to the examples, here is an immediate theoretical consequence for group actions (which is in fact related to the last two bullet-points):

II.N.5. COROLLARY. *Given a finite group $G$ acting transitively on a finite set* X (*with at least* 2 *elements*), *there exists an element $g \in G$ which acts without fixed points.*[32] *In fact, there are at least $\frac{|G|}{|X|}$ such elements.*

PROOF. We can consider the actions of $G$ on X, and also on $X \times X$ by $g.(x, x') := (g.x, g.x')$. Write $\chi(g) := |X^g|$, so that also $\chi^2(g) = |(X \times X)^g|$; and for any function $f$ on $G$ and subset $S \subset G$, write $\int_S f := \frac{1}{|G|}\sum_{g\in S} f(g)$. We want to show that the subset of fixed-point-free elements $G_0 := \{g \in G \mid \chi(g) = 0\} \subset G$ has $C := \frac{|G_0|}{|G|} = \int_{G_0} 1 \geq \frac{1}{|X|}$.

Burnside plus transitivity tell us that

$$1 = |X/G| = \int_G \chi \quad \text{and} \quad 2 \leq |(X \times X)/G| = \int_G \chi^2, \tag{II.N.6}$$

[32]Up to this point, this Corollary is a theorem of Jordan from 1872. The last sentence is due to Cameron-Cohen (1992) and its proof to Serre (2003).

as the "diagonal" $\Delta_X := \{(x,x) \mid x \in X\} \subset X \times X$ is an orbit. If $g \notin G_0$, then $1 \leq \chi(g) \leq |X|$; and so $\int_{G \setminus G_0} (\chi(g)-1)(\chi(g)-|X|) \leq 0$, which can be rewritten as

$$\int_G (\chi(g)-1)(\chi(g)-|X|) \leq \int_{G_0} (\chi(g)-1)(\chi(g)-|X|) = C|X|.$$

By (II.N.6), the LHS becomes

$$\int_G \chi^2 \; - \; (1+|X|) \int_G \chi \; + \; |X| \int_G 1 \; \geq \; 2-(1+|X|)+|X| = 1.$$

Conclude that $C \geq \frac{1}{|X|}$ as desired. $\square$

Here is a straightforward application of Burnside to a counting problem.

II.N.7. EXAMPLE. *How many inequivalent bracelets with five beads can you make with only black and white beads?*

Here two bracelets are *equivalent* if they are the same after rotating and flipping them, i.e. if they belong to the same orbit under $D_5$. So we take X to be the set of B/W colorings of the sides of a regular 5-gon, and $G := D_5$ (acting in the obvious way). We arrive at the table

| type of element of $D_5$ | # of such | # of fixed points $\lvert X^g \rvert$ |
|---|---|---|
| 1 (identity) | 1 | $2^5$ $(= \lvert X \rvert)$ |
| $hr^k$ (flip) | 5 | $2^3$ (edges flipped into each other must be same color in order to be fixed) |
| $r^k$ (rotation) | 4 | 2 (all edges must be same color to yield a "fixed point") |

The number of bracelets (i.e. orbits) is then simply

$$|X/G| = \frac{1}{|D_5|} \sum_{g \in D_5} |X^g| = \frac{1}{10}\{1 \cdot 2^5 + 5 \cdot 2^3 + 4 \cdot 2\} = 8.$$

In the HW, you will determine (up to rotational symmetry) *how many different ways one can paint the edges of a tetrahedron red, green, or blue.*

**Crystallographic groups.** We now turn to a much more interesting example. By a *lattice* in $\mathbb{R}^3$, we mean a subgroup of $(\mathbb{R}^3, +, 0)$ isomorphic to $\mathbb{Z}^3$. Denote by $\mathrm{O}(3)$ the *orthogonal group* comprising $3 \times 3$ matrices $A$ with real entries and ${}^tAA = \mathrm{I}_3$. The determinant defines a surjective homomorphism $\det\colon \mathrm{O}(3) \twoheadrightarrow \{\pm 1\}$, with kernel the *special orthogonal group* $\mathrm{SO}(3)$. Heuristically, this is the group of rotations about the origin in $\mathbb{R}^3$, while $\mathrm{O}(3)$ also includes reflections and $-\mathrm{I}_3$.

II.N.8. DEFINITION. A *crystallographic group* is a (nontrivial) finite subgroup $G \leq \mathrm{SO}(3)$ or $\mathrm{O}(3)$ that preserves a lattice in $\mathbb{R}^3$.

The mathematical determination (Hessel 1830) of the existence of exactly 32 geometric crystal classes, essentially via such groups, predated the actual ability to look inside crystals with X-rays by something like 80 years. We shall restrict ourselves here to the rotational case, so that we can obtain a complete classification using Burnside.

So let $G \leq \mathrm{SO}(3)$ be a finite subgroup. Each $g \in G \backslash \{1\}$ is a rotation about some axis $\ell_g$, and we set $\{p_g, p'_g\} := \ell_g \cap S^2$, where $S^2$ is the unit 2-sphere centered at the origin. Let

$$\mathrm{X} := \bigcup_{g \in G \backslash \{1\}} \{p_g, p'_g\}.$$

II.N.9. CLAIM. *$G$ acts on* X.

PROOF. We first clarify what we mean by this. Since $\mathrm{SO}(3)$ acts (by matrix multiplication) on $\mathbb{R}^3$, and in fact on $S^2$,[33] we need only check that X is *closed* under the action of $G$. Given $x \in \mathrm{X}$, $x = p_{g_0}$ (or $p'_{g_0}$) for some $g_0 \in G$. That is, $g_0 x = x$.

Consider $g.x = gx$ (where the RHS means the matrix $g \in G$ times the vector $x$). This is in X $\iff$ it is $p_{g_1}$ (or $p'_{g_1}$) for some $g_1 \in G$. But

$$g g_0 g^{-1}.gx = g g_0.x = g.g_0 x = g.x = gx,$$

and so $g_1 := g g_0 g^{-1}$ works. $\square$

[33] Given a vector $\vec{v} \in \mathbb{R}^3$ and matrix $A \in \mathrm{SO}(3)$, we have that $A\vec{v} \cdot A\vec{v} = {}^t\vec{v}\,{}^tAA\vec{v} = {}^t\vec{v}\vec{v} = \vec{v} \cdot \vec{v} \implies A$ preserves the length of $\vec{v}$. Thus $S^2$ is closed under the action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$.

Now let $n = |G|$ and $r = |X/G|$. Choose representatives $x_1, \ldots, x_r$ in each orbit and put $n_i := |G_{x_i}|\,(\leq n)$. By the orbit-stabilizer theorem, the orbit sizes are $|G(x_i)| = \frac{n}{n_i}$, so that

$$|X| = \sum_{i=1}^{r} \frac{n}{n_i}.$$

Burnside yields $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g| \implies$

$$\begin{aligned} r &= \tfrac{1}{n}\left(|X| + \textstyle\sum_{g\in G\setminus\{1\}}|\{p_g, p'_g\}|\right) \\ &= \tfrac{1}{n}\left(\textstyle\sum_{i=1}^{r}\frac{n}{n_i} + (n-1)2\right) \\ &= \textstyle\sum_{i=1}^{r}\frac{1}{n_i} \ + \ 2 - \frac{2}{n}. \end{aligned}$$

For each $x \in X$, $|G_x| \geq 2$ by definition (there is some non-identity element stabilizing it)

$$\begin{aligned} &\implies \text{ each } \ n_i \geq 2 \\ &\implies r \leq \tfrac{r}{2} + 2 - \tfrac{2}{n} \\ &\implies \tfrac{r}{2} \leq 2 - \tfrac{2}{n} < 2, \end{aligned}$$

so $r < 4$. If $r = 1$ then $\frac{2}{n} = \frac{1}{n_1} + 1 \geq \frac{1}{n} + 1 \implies \frac{1}{n} \geq 1$, which is absurd.

$$\implies r = 2 \ \text{ or } \ 3.$$

<u>**Case r = 2**</u>: $\quad 2 = \frac{1}{n_1} + \frac{1}{n_2} + 2 - \frac{2}{n}$

$$\begin{aligned} &\implies \tfrac{2}{n} = \tfrac{1}{n_1} + \tfrac{1}{n_2} \geq \tfrac{2}{n} \\ &\implies \tfrac{2}{n} = \tfrac{1}{n_1} + \tfrac{1}{n_2} \quad (\text{and } \ n_1, n_2 \leq n) \\ &\implies n_1 = n = n_2 \\ &\implies \text{every } g \in G \text{ stabilizes every } x \in X. \end{aligned}$$

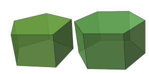
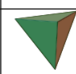
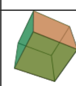
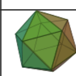
The only way that even one non-identity element of $G$ stabilizes every $x \in X$, is if there are only 2 points $p, p' \in X$; and then the elements of $G$ are rotations about the axis they span (by multiples of $\frac{2\pi}{n}$)

$$\implies G \cong \mathbb{Z}_n.$$

__Case r = 3__: $(1 <) \frac{2}{n} + 1 \overset{(*)}{=} \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}$. We may assume $2 \leq n_1 \leq n_2 \leq n_3 \leq n$. Now

- if $n_1 > 2$ then RHS$(*) \leq 1$, which is impossible; so $n_1 = 2$.
- if $n_2 \geq 4$ then RHS$(*) \leq 1$, again impossible; so $n_2 = 2$ or $3$.
- if $n_2 = 3$ then for the same reason, $n_3 < 6$.

Thus our options are confined to the left-hand column of the following table:

| $(n_1, n_2, n_3)$ | $n\,(= \lvert G \rvert)$ | $G$ | geometric realization: rotational symmetries of ... |
|---|---|---|---|
| $(2,2,k)_{k\geq 2}$ | $2k$ | $D_k$ |  ... prism on regular $k$-gon |
| $(2,3,3)$ | 12 | $\mathfrak{A}_4$ |  tetrahedron |
| $(2,3,4)$ | 24 | $\mathfrak{S}_4$ |  cube |
| $(2,3,5)$ | 60 | $\mathfrak{A}_5$ |  icosahedron |

Fix an orbit $G(x_i) \subset \mathrm{X}$ ($i = 1, 2,$ or $3$). The stabilizer $G_{x_i}$ of the representative $x_i$ comprises all the elements of $G$ which are rotations about $x_i$ (viewed as a vector in $\mathbb{R}^3$). So $n_i = |G_{x_i}|$ is the order of this axis of rotation, and also of the other axes of rotation in $G(x_i)$, of which there are $\frac{1}{2}|G(x_i)| = \frac{n}{2n_i}$. Two examples:[34]

- $(2,2,6)$ corresponds to the prism on the regular hexagon shown. The orbits have sizes $6, 6, 2$, corresponding to 3 axes (with $180°$ rotation, through the vertical edges), 3 axes (through the vertical faces, again with $180°$ rotation), and 1 vertical axis (with $60°$ rotation).
- $(2,3,3)$ corresponds to the tetrahedron. The orbits have sizes $6, 4, 4$, so there are 3 axes (through pairs of nonintersecting edges) of $180°$-rotation, and $2 + 2 = 4$ axes (through each vertex and the opposite face) of $120°$-rotation.

[34]For the cube, see II.F.4(v).

This gives a flavor of how one derives the table.

We have at this point finished the classification of finite rotational *symmetry groups* in space.

Next, impose the condition that we are not just stabilizing a single "crystal unit" but a *pattern* which may be continued infinitely to fill up $\mathbb{R}^3$. So we ask for $G$ to act on a lattice $\Lambda \subset \mathbb{R}^3$, comprising all $\mathbb{Z}$-linear combinations of three linearly independent vectors $\vec{u}, \vec{v}, \vec{w}$. Now:

- If we think of $g \in G$ as a $3 \times 3$ matrix written with respect to the basis $\vec{u}, \vec{v}, \vec{w}$, then the trace $\mathrm{tr}(g) \in \mathbb{Z}$ since $g$ will have integer entries.
- If we write $g$ instead with respect to an orthonormal basis including $p_g$ (a vector along the axis of rotation), it takes the form
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}$$
and so $\mathrm{tr}(g) = 2\cos(\theta) + 1$.
- From matrix algebra, you know that the trace of $g$ is independent of the choice of basis. Hence, we must have $2\cos(\theta) \in \mathbb{Z}$. Since $g$ is a finite-order rotation, $\theta = \frac{2\pi}{k}$, for some $k \in \mathbb{Z}$. But $2\cos(\frac{2\pi}{k}) \in \mathbb{Z}$ only for $k = 1, 2, 3, 4, 6$.

Therefore, for a *crystallographic* group $G \leq \mathrm{SO}(3)$, we must have that all of the $n_i$ belong to $\{1, 2, 3, 4, 6\}$. Throwing out all other groups in our list, we are left with *10 nontrivial rotational crystallographic groups*:

$$\mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_6,\ D_2(\cong V_4)\ , D_3(\cong \mathfrak{S}_3)\ , D_4,\ D_6,\ \mathfrak{A}_4, \text{ and } \mathfrak{S}_4.$$