V. Remarks on Associative Algebras

V.A. Algebras over a field

Let $\mathbb{F}$ be a field. What would you call an $\mathbb{F}$-vector space where you can multiply vectors?

V.A.1. DEFINITION. An **$\mathbb{F}$-algebra** (or **algebra over** $\mathbb{F}$) is a ring A, together with a scalar multiplication by $\mathbb{F}$ which makes A into an $\mathbb{F}$-vector space and satisfies

$$f \cdot (a_1 a_2) = (f \cdot a_1) a_2 = a_1 (f \cdot a_2). \tag{V.A.2}$$

V.A.3. EQUIVALENT DEFINITION. A ring A together with an embedding $\varepsilon \colon \mathbb{F} \hookrightarrow C(A)$.

PROOF THAT V.A.1 IMPLIES V.A.3. Let A be an $\mathbb{F}$-algebra, and (for each $f \in \mathbb{F}$) set $\varepsilon(f) := f \cdot 1_A \in A$. Then:

- ε is a homomorphism (from $\mathbb{F}$ to A): since A is a vector space over $\mathbb{F}$, we have $(f_1 + f_2) \cdot a = f_1 \cdot a + f_2 \cdot a$ and $(f_1 f_2) \cdot a = f_1 \cdot (f_2 \cdot a)$, and setting $a = 1_A$ (and using (V.A.2)) does the job;
- ε is injective because $\mathbb{F}$ is a field; and
- $\varepsilon(f) \in C(A)$ because $a\varepsilon(f) = a(f \cdot 1_A) = f \cdot (a 1_A) = f \cdot (1_A a) = (f \cdot 1_A) a = \varepsilon(f) a$.

[The other direction is left to you.] □

We will now stop writing the "$\cdot$". Also, notice that ε embeds $\mathbb{F}$ as a subring of A whose elements commute with everything; so we can identify $\mathbb{F}$ with this subring, and drop "ε".

V.A.4. EXAMPLES. (i) **Field extensions** $\mathbb{E}/\mathbb{F}$: these are, by definition, fields containing $\mathbb{F}$.
(ii) Polynomial algebras $\mathbb{F}[x_1, \ldots, x_n]$.

(iii) Product algebras $\mathbb{F} \times \cdots \times \mathbb{F}$.
(iv) Matrix algebras $M_n(\mathbb{F})$ (= the ring of endomorphisms of an n-dimensional $\mathbb{F}$-vector space).
(v) Group algebra $\mathbb{F}[G]$ (of a finite group G).
(vi) Ring of endomorphisms of an $\mathbb{F}[\lambda]$-module.
(vii) Quaternion algebras (e.g., $\mathbb{H}$ as an $\mathbb{R}$-algebra; or the rational quaternion algebras from HW 6 #6).
(viii) Exterior algebras (defined below).

V.A.5. DEFINITION. $A' \subset A$ is an **$\mathbb{F}$-subalgebra** if A' is a sub-$\mathbb{F}$-vector space *and* subring. The $\mathbb{F}$-subalgebra generated by a set $\mathcal{S} \subset A$ is

$$\mathbb{F}[\mathcal{S}] \;:=\; \bigcap_{\substack{A' \subset A \text{ subalg.} \\ A' \supset \mathcal{S}}} A' \;=\; \left\{ \begin{matrix} \text{elements of } A \text{ that can be written} \\ \text{as polynomials over } \mathbb{F} \text{ in } \{1\} \cup \mathcal{S} \end{matrix} \right\}.$$

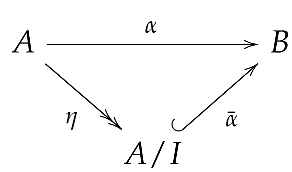
(So there is an obvious notion of finitely generated $\mathbb{F}$-algebra; this is much weaker than finite-*dimensionality* of A as vector space over $\mathbb{F}$.)

V.A.6. DEFINITION. Officially, $I \subset A$ is an **(algebra) ideal** if I is an ideal in the ring A *which is an $\mathbb{F}$-vector subspace*. But in point of fact, since $\mathbb{F} \subset A$, any ring-ideal of A is already closed under multiplication by $\mathbb{F}$, hence also an algebra-ideal; so there's no difference.

Given an ideal $I \subset A$, the quotient A/I has an $\mathbb{F}$-algebra structure: the composition $\mathbb{F} \hookrightarrow A \twoheadrightarrow A/I$ is still injective by (III.F.1).

V.A.7. DEFINITION. A map $\alpha \colon A \to B$ of $\mathbb{F}$-algebras is an **$\mathbb{F}$-algebra homomorphism** if it is a ring homomorphism which is $\mathbb{F}$-linear (i.e., $\alpha(fa) = f\alpha(a)$ for all $f \in \mathbb{F}$ and $a \in A$).

As usual, we get a "Fundamental Theorem" to the effect that $I := \ker(\alpha)$ is an (algebra) ideal and there exists $\bar{\alpha}$ such that



commutes.

There is also an algebro-theoretic version of Cayley's theorem:

V.A.8. THEOREM. *Any $\mathbb{F}$-algebra A is isomorphic to a subalgebra of an algebra of endomorphisms of a vector space.*

PROOF. Consider A as an $\mathbb{F}$-vector space, and map

$$\begin{aligned} \ell\colon A &\longrightarrow \mathrm{End}_{\mathbb{F}}(A) \\ a &\longmapsto \ell_a := \text{ left-multiplication by } a. \end{aligned}$$

Since A is an algebra,

$$\ell_a(f\alpha) = a(f\alpha) = f(a\alpha) = f\ell_a(\alpha)$$

$\implies \ell_a \in \mathrm{End}_{\mathbb{F}}(A)$. Moreover, we know that ℓ is an injective ring homomorphism.[1] Finally, ℓ is a homomorphism of $\mathbb{F}$-vector spaces, since by (V.A.2) we have $\ell_{fa}(\alpha) = (fa)(\alpha) = f(a\alpha) = f\ell_a(\alpha)$ and thus $\ell_{fa} = f\ell_a$. □

Exterior algebras. For this extended example, start with a vector space $V/\mathbb{F}$ of dimension n (without a multiplication law, of course). We would like an algebra A generated by V such that

(V.A.9) $$v^2 = 0 \qquad (\forall v \in V).$$

Let $\{u_1, \ldots, u_n\} \subset V$ be a basis. Then (V.A.9) gives

$$0 = (u_i + u_j)^2 - u_i^2 - u_j^2 = u_i u_j + u_j u_i \quad \implies$$

(V.A.10) $$u_i u_j = -u_j u_i.$$

If we take $i_1 < \cdots < i_k$, then this yields

(V.A.11) $$u_{i_{\sigma(1)}} \cdots u_{i_{\sigma(k)}} = \mathrm{sgn}(\sigma) u_{i_1} \cdots u_{i_k} \qquad (\forall \sigma \in \mathfrak{S}_k)$$

since $\mathrm{sgn}(\sigma) = (-1)^{\#\text{ of transpositions in }\sigma}$ and σ can be built from *adjacent* transpositions (as in (V.A.10)). Henceforth, we shall write "$\wedge$" for our product and make the formal

[1] Why? Recall $\ell_a = 0 \implies 0 = \ell_a 1 = a \cdot 1 = a$.

V.A.12. DEFINITION. $\bigwedge_{\mathbb{F}}^{\bullet} V$ is the $\mathbb{F}$-algebra with

- ($\mathbb{F}$-vector space) basis consisting of *monomials*[2]

$$u_{\mathcal{I}} := u_{i_1} \wedge \cdots \wedge u_{i_k} \qquad (i_1 < \cdots < i_k)$$

  where $\mathcal{I} = \{i_1, \ldots, i_k\}$ ranges over subsets of $\{1, \ldots, n\}$,
- product

$$u_{\mathcal{I}} \wedge u_{\mathcal{J}} = \begin{cases} 0, & \text{if } \mathcal{I} \cap \mathcal{J} \neq \emptyset \\ \operatorname{sgn}(\sigma_{\mathcal{I}\mathcal{J}}) u_{\mathcal{I}\cup\mathcal{J}}, & \text{if } \mathcal{I} \cap \mathcal{J} = \emptyset \end{cases}$$

  where $\sigma_{\mathcal{I}\mathcal{J}}$ shuffles $\mathcal{I}$ and $\mathcal{J}$ together, and
- identity $u_{\emptyset} = 1$.

We have $\dim_{\mathbb{F}}(\bigwedge_{\mathbb{F}}^{\bullet} V) = \sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$, and

$$\bigwedge\nolimits_{\mathbb{F}}^{\bullet} V = \bigoplus\nolimits_k \bigwedge\nolimits_{\mathbb{F}}^{k} V, \tag{V.A.13}$$

where $\bigwedge_{\mathbb{F}}^{k} V$ is the subspace spanned by monomials of degree k.

V.A.14. EXAMPLE. We illustrate the product: taking $\mathcal{I} = \{1, 3, 6\}$ and $\mathcal{J} = \{2, 5\}$, we "shuffle" them together by jumping 2 over 3 and 6, then 5 over 6, for a total of three transpositions. Hence

$$(\underbrace{u_1 \wedge u_3 \wedge u_6}_{u_{\mathcal{I}}}) \wedge (\underbrace{u_2 \wedge u_5}_{u_{\mathcal{J}}}) = (-1)^3 u_1 \wedge u_2 \wedge u_3 \wedge u_5 \wedge u_6 = -u_{\mathcal{I}\cup\mathcal{J}}.$$

V.A.15. PROPOSITION. *Let $B = (b_{ij}) \in M_n(\mathbb{F})$. Then*

$$(b_{11}u_1 + \cdots + b_{n1}u_n) \wedge \cdots \wedge (b_{n1}u_1 + \cdots + b_{nn}u_n) = (\det(B))\, u_1 \wedge \cdots \wedge u_n.$$

PROOF. Expanding the LHS gives

$$\textstyle\sum_{i_1,\ldots,i_n} (b_{i_1,1} \cdots b_{i_n,n}) u_{i_1} \wedge \cdots \wedge u_{i_n}.$$

Since $v \wedge v = 0$, terms with $i_j = i_k$ for $j \neq k$ vanish, and this becomes

$$\textstyle\sum_{\sigma\in\mathfrak{S}_n} (b_{\sigma(1),1} \cdots b_{\sigma(n),n}) u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(n)}$$
$$= \left(\textstyle\sum_{\sigma} (\prod_i b_{\sigma(i),i}) \operatorname{sgn}(\sigma)\right)\, u_1 \wedge \cdots \wedge u_n,$$

where the parenthetical quantity is just $\det(B)$. □

[2]The *degree* of the monomial is $k = |\mathcal{I}|$.

V.A.16. THEOREM. *Assume* $|\mathbb{F}| = \infty$, *let* $Q \in \mathbb{F}[x_{11}, x_{12}, \ldots, x_{nn}] =: \mathbb{F}[\{x_{ij}\}]$ *be a homogeneous polynomial of degree* q *in* n^2 *variables, and define* $Q(B) := Q(\{b_{ij}\})$ *for matrices* $B \in M_n(\mathbb{F})$. *Assume that* $Q(\mathbb{1}_n) = 1$ *and* $Q(BB') = Q(B)Q(B')$. *Then* Q *is a power of* det.

SKETCH. Since Q is homogeneous,

(V.A.17)

$$Q(B)Q(\text{adj}B) = Q((\det B)\mathbb{1}_n) = (\det B)^q Q(\mathbb{1}_n) = (\det B)^q.$$

Writing $X = (x_{ij})$, set $P(X) := Q(X)Q(\text{adj}X) - (\det X)^q \in \mathbb{F}[\{x_{ij}\}]$. Since nontrivial polynomials over an infinite field do not evaluate to zero everywhere, but $P(B) = 0$ for all B by (V.A.17), we must have $P = 0$. Hence $Q(X)Q(\text{adj}X) = (\det X)^q$ in $\mathbb{F}[\{x_{ij}\}]$ and so $Q(X) \mid (\det X)^q$. But $\mathbb{F}[\{x_{ij}\}]$ is a UFD, and so the result is clear if we know that $\det X$ is irreducible in $\mathbb{F}[\{x_{ij}\}]$. This is proved in **[Jacobson]**. ☐

Exterior algebras are ubiquitous in algebra (esp. representation theory) and geometry (differential forms).