Problem Set 1

Hand in problems 5-10. Make sure you can do 3 and 4; 1 and 2 should be regarded as optional but enlightening.

(1) Show that if p is a prime and $p \mid ab$ then either $p \mid a$ or $p \mid b$.
(2) Prove that Zorn's lemma implies the axiom of choice.
(3) Check that Example (I.A.2)(iv) in the notes is transitive (hence an equivalence relation), and prove that the quotient set is isomorphic (as a set — i.e. there exists a bijective mapping) to $\mathbb{Z}$.
(4) Let X be a set. Prove that the set of partitions of X is isomorphic to the set of equivalence relations on X.
(5) Referring to Example (II.C.3)(ii) in the notes, write down (with some sort of proof) the relations for the dihedral group D_n. [Hint: look at Jacobson p. 36 #8.] Write down the multiplication tables for D_3 and $\mathfrak{S}_3$ (symmetric group).
(6) Let $X := \{1, \alpha, \beta\}$ (3-element set). There are 3^4 ways to define a binary operation on X which satisfies (II.A.1)(ii) (best visualized in table form). How many give a monoid and how many give a group? In each case, are there any nonabelian examples?
(7) Find a group structure on the set of integer solutions to $x^2 - 5y^2 = \pm 4$ which makes it into an abelian group isomorphic to $\mathbb{Z} \times \{1, -1\}$. [Hint: consider numbers of the form $\frac{x+y\sqrt{5}}{2}$.] What do you notice about the y-values in this set?
(8) Write $(456)(567)(671)(123)(234)(345)$ as a product of disjoint cycles; also determine its sign.
(9) Show that any finite group of even order contains an element $a \neq 1$ such that $a^2 = 1$.
(10) Show that a group G cannot be a union of two proper subgroups.