PROBLEM SET 10

(Hand in all.) Below, $d(\neq 0,1)$ is always squarefree.

(1) Let $K = \mathbb{Q}[\sqrt{-29}]$. From III.L.26 we know that $[\mathfrak{P}_5] \in Cl(K)$ has order 3, and we also note that $(2) = (2, 1+\sqrt{-29})^2 =: \mathfrak{P}_2^2$. Show that $\mathcal{O}_K$ has ideals of norm 3 and 11 of order 6 in $Cl(K)$. [Hint: start by looking for principal ideals of norm 30 and 33.]

(2) We explained how odd primes p decompose in number rings. What about the even prime? Let $K = \mathbb{Q}[\sqrt{d}]$, and show that (in $\mathcal{O}_K$)

$$\begin{array}{lcc} d \underset{(4)}{\equiv} 2 & \Longrightarrow & (2) = (2, \sqrt{d})^2 \\ d \underset{(4)}{\equiv} 3 & \Longrightarrow & (2) = (2, 1+\sqrt{d})^2 \\ d \underset{(8)}{\equiv} 1 & \Longrightarrow & (2) = (2, \frac{1+\sqrt{d}}{2})(2, \frac{1-\sqrt{d}}{2}) \\ d \underset{(8)}{\equiv} 5 & \Longrightarrow & (2) \text{ prime} \end{array}$$

(3) Let $K = \mathbb{Q}[\sqrt{-26}]$. Find all non-principal ideals of norm 30 in $\mathcal{O}_K$. [Hint: here are some of your tools: Prop. III.L.25, Pell's equation (i.e. using solutions of $x^2 + 26y^2 = m$ to test whether there exists a principal ideal of norm m), uniqueness of ideal factorization, and Caesar.]

(4) Show that $X^2 = Y^3 - 14$ has no solution with $X, Y \in \mathbb{Z}$. You may assume that $h_{\mathbb{Q}(\sqrt{-14})} = 4$. [Hint: if (X,Y) is a solution, put $\alpha := X + \sqrt{-14}$ (not $X + Y\sqrt{-14}$!!). Turn the equation into an equation of ideals, decompose both sides into prime factors, and use uniqueness of ideal factorization to deduce that α is a cube in $\mathbb{Z}[\sqrt{-14}]$.]

(5) Let $K = \mathbb{Q}[\sqrt{d}]$, $d \underset{(4)}{\equiv} 2$ or 3, and $I \subset \mathcal{O}_K$ an ideal. Show that $\mathfrak{N}(I) = |\mathcal{O}_K/I|$, where $\mathfrak{N}(I)$ is defined via Hurwitz. [Hint: first compute $|\mathcal{O}_K/I|$ as a determinant.]