Problem Set 4

Hand in all.

(1) [Jacobson p. 58 #6] Let G_1 and G_2 be simple groups. Show that every normal subgroup of $G = G_1 \times G_2$ other that G and $\{1\}$ is isomorphic to either G_1 or G_2.

(2) If $HK = G$ for $H, K \leq G$, then $|H||K| = |G||H \cap K|$. [See Jacobson p. 58 #9.]

(3) Compute the automorphism groups $\mathrm{Aut}(G)$, $\mathrm{Inn}(G)$, and $\mathrm{Out}(G)$ for $G = D_4$. [Hint: try labeling automorphisms by where they send r and h.]

(4) [Jacobson p. 70 #1] Let S be a subset of a group G such that $g^{-1}Sg \subset S$ for any $g \in G$. Show that the subgroup $\langle S \rangle$ generated by S is normal. Let T be any subset of G and let $S = \cup_{g \in G} g^{-1}Tg$. Show that $\langle S \rangle$ is the normal subgroup generated by T (i.e. the smallest normal subgroup [or equivalently intersection of all normal subgroups] containing T).

(5) [Jacobson p. 71 #3] Using the generators (12), (13), ..., $(1n)$ for $\mathfrak{S}_n$, show that $\mathfrak{S}_n$ is defined by the following relations on $x_1, x_2, \ldots, x_{n-1}$ in $\mathscr{F}_{n-1}$: x_i^2, $(x_i x_j)^3$, $(x_i x_j x_i x_k)^2$, where i, j, k are distinct. [Hint: use induction on n, and the (compatible) inclusions $\mathscr{F}_{n-2} \hookrightarrow \mathscr{F}_{n-1}$ and $\mathfrak{S}_{n-1} \hookrightarrow \mathfrak{S}_n$.]

(6) Construct explicitly a homomorphism Φ from the quaternions Q to the Klein 4-group V_4 with kernel $\{\pm 1\}$.

(7) Recall that the commutator subgroup $[G, G] \leq G$ is the subgroup generated by all commutators $[g_1, g_2] := g_1^{-1} g_2^{-1} g_1 g_2$ $(g_1, g_2 \in G)$. Show (i) that (for an arbitrary group G) $[G, G] \trianglelefteq G$, and (ii) that the resulting quotient group $G/[G, G]$ is abelian. Finally, (iii) show that $\mathscr{A}_S \cong \mathscr{F}_S / [\mathscr{F}_S, \mathscr{F}_S]$ for any set S.

(8) Check the "Claim" in the proof of II.J.5 of the notes: given an automorphism $\alpha \in \mathrm{Aut}(\mathfrak{S}_n)$ sending transpositions to transpositions and $(12) \mapsto (ab)$, $(13) \mapsto (ac)$ $(c \neq a, b)$ in particular, show that $\alpha((1y)) = (ad)$ for some $d \neq a$ (depending on $y \neq 1$). [Warning: you may not use any results about automorphisms of $\mathfrak{S}_n$ beyond, say, what has been shown up to that point in the proof. The argument should be in the spirit of what has come before, and is short.]