Problem Set 5

(Hand in all.)

(1) Determine all subgroups of the mod 3 Heisenberg group (II.M.17). For the normal ones, determine the quotient group.
(2) Show that $\mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_p) \cong \mathrm{GL}(2, \mathbb{Z}_p)$ (2×2 matrices with mod p entries which are invertible mod p), and check that this has order $(p-1)^2 p(p+1)$.
(3) Let $\theta \colon H \to \mathrm{Aut}(K)$ be a group homomorphism. Prove that $K \rtimes H$ (one usually writes this instead of $\rtimes_\theta$) sits in a *split* short exact sequence with K on the left and H on the right.
(4) Classify the groups of order 490 up to isomorphism. Start by proving that none of them are simple. [Edit: this problem is a bit too hard I think. More reasonable is this: classify the groups of order 245 up to isomorphism. Now, what can you say *about* the classification of groups of order 490? You don't have to do a complete classification, but say something about it and attempt a partial classification.]
(5) Find all Sylow 3-subgroups of the symmetric group $\mathfrak{S}_6$.
(6) Do Jacobson p. 79 #10. (by a famous theorem, these wreath products $G \wr H$ actually turn out to contain all extensions of H by G as subgroups. You may also want to look at #16 on p. 84, but you don't have to hand it in.