Problem Set 6

(Hand in all.)

(1) Up to rotational symmetry, how many different ways can you paint the edges of a tetrahedron red, green, or blue?

(2) Give a short proof (without Burnside) of the result that a finite group G acting transitively on a finite set X (with at least 2 elements) has at least one $g \in G$ that acts without fixed points. [Hint: consider the union of the stabilizers G_x, and don't forget that transitivity means that X is one big orbit.]

(3) [Jacobson p. 91 #2] Show that a domain contains no idempotents ($e^2 = e$) except $e = 0$ and $e = 1$. An element z is called *nilpotent* if $z^n = 0$ for some $n \in \mathbb{Z}_{>0}$. Show that 0 is the only nilpotent in a domain.

(4) [Jacobson p. 91 #6] Let u be an element of a ring that has a right inverse. Prove that the following conditions on u are equivelent: (1) u has more than one right inverse; (2) u is not a unit; (3) u is a left zero-divisor.

(5) [Jacobson p. 91 #7] Prove that if an element of a ring has more than one right inverse then it has infinitely many. Construct a counterexample to show that this does not hold for monoids.

(6) [Jacobson p. 100 #7] Let m and n be non-zero integers and let r be the subset of $M_2(\mathbb{C})$ consisting of the matrices of the form

$$\begin{pmatrix} a + b\sqrt{m} & c + d\sqrt{m} \\ n(c - d\sqrt{m}) & a - b\sqrt{m} \end{pmatrix}$$

where $a, b, c, d \in \mathbb{Q}$. Show that R is a subring of $M_2(\mathbb{C})$ and that R is a division ring if and only if the rational numbers x, y, z, t satisfying the equation $x^2 - my^2 - nz^2 + mnt^2 = 0$ are $x = y = z = t = 0$. Give a choice of m, n for which r is a division ring and a choice of m, n for which R is not a division ring. [N.B. These rings are called "rational quaternion algebras".]

(7) Find the group of units in all number rings $\mathbb{Z}[\sqrt{d}]$ for integers $d < 0$ and $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ for $d < 0$ and $d \underset{(4)}{\equiv} 1$.

(8) Show that $R = \mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$ is Euclidean.