PROBLEM SET 8

(Hand in all.)

(1) [Jacobson p. 111 #17] Determine the ideals and maximal ideals and prime ideals of $\mathbb{Z}_{60} \cong \mathbb{Z}/(60)$.

(2) [Jacobson p. 118 #5] Let R be a commutative ring, and S a submonoid of the multiplicative monoid of R. In $R \times S$, define $(a,s) \sim (b,t)$ if there exists a $u \in S$ such that $u(at - bs) = 0$. Show that this is an equivalence relation in $R \times S$. Denote the equivalence class of (a,s) as a/s and the quotient set consisting of these classes as RS^{-1}. Show that RS^{-1} becomes a ring relative to $a/s + b/t = (at+bs)/st$, $(a/s)(b/t) = ab/st$, $0 = 0/1$, and $1 = 1/1$. Show that $a \mapsto a/1$ is a homomorphism of R into RS^{-1}, and that this is a monomorphism if and only if no element of S is a zero-divisor in R. Show that the elements $s/1$, $s \in S$, are units in RS^{-1}.

(3) [Jacobson p. 126 #2] Show that $\sqrt{3} \notin \mathbb{Q}[\sqrt{2}]$ and that the real numbers $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$ are linearly independent over $\mathbb{Q}$. Show that $u = \sqrt{2} + \sqrt{3}$ is algebraic and determine an ideal I such that $\mathbb{Q}[x]/I \cong \mathbb{Q}[u]$.

(4) [Jacobson p. 133 #9] Show that the ideal $(3, x^3 - x^2 + 2x - 1)$ in $\mathbb{Z}[x]$ is not principal.

(5) [Jacobson p. 133 #13] Prove Wilson's theorem: if p is a prime in $\mathbb{Z}$, then we have $(p-1)! \underset{(p)}{\equiv} -1$.

(6) A commutative ring is *local* if it has a unique maximal ideal.

(a) Using problem (2) above with $R = \mathbb{Z}$, $S = \{b \in \mathbb{Z} \mid p \nmid b\}$, prove that RS^{-1} is local.

(b) If $\mathcal{R}$ is a local ring with unique maximal ideal $\mathfrak{m}$, prove that $a \in \mathcal{R}$ is a unit if and only if $a \notin \mathfrak{m}$. [Hint: you will need to use (III.D.27).]

(7) Find an inverse (fractional ideal) for $I = (15 + 25\sqrt{-10},\ 25 - 40\sqrt{-10}) \subset \mathbb{Z}[\sqrt{-10}] = R$.

(8) [Jacobson p. 140 #3] (Newton's identities. See Jacobson. Statement of problem is long but solution need not be.)