I.F. Separable, normal, and Galois extensions

At the heart of Galois theory is a correspondence between intermediate fields in an extension and subgroups of its automorphism group. This takes its nicest form for *Galois extensions*, where the automorphism group has order equal to the degree of the extension. This setting is obtained by imposing on our extensions the *separability* and *normality* conditions, alluded to in §I.D, which we now explain in detail.

Separable extensions.

With the understanding (from §I.E) that inseparable polynomials only occur *over* non-perfect fields K, we now make the

I.F.1. DEFINITION. (i) Let L/K be an extension, and $\alpha \in L$. We say α is **separable** over K if α is algebraic over K and its minimal polynomial m_α over K is separable.

(ii) An extension L/K is **separable** if it is algebraic and each $\alpha \in L$ is separable over K.

I.F.2. PROPOSITION. *Intermediate extensions in separable extensions are separable.*

PROOF. Let L/K be separable, and M be a subfield of L containing K. Obviously M/K is separable; the issue is L/M.

Take $\alpha \in L$, and let m_α and p_α denote its minimal polynomials over K resp. M. Then $p_\alpha \mid m_\alpha$ in $M[x]$. Let N/M be a splitting field extension for m_α. Since m_α is separable over K, and N contains a splitting field for m_α over K, we have $m_\alpha(x) = \prod_i (x - \alpha_i)$ in $N[x]$, with $\alpha_i \in N$ distinct. Hence $p_\alpha(x) = \prod_\ell (x - \alpha_{i_\ell})$ in $N[x]$, with $\alpha_{i_\ell} \in N$ distinct. Conclude that p_α is separable. □

I.F.3. THEOREM. *Let M/K be an extension of degree $d := [M{:}K]$, and fix an embedding $\jmath\colon K \hookrightarrow L$. Write r for the number of embeddings $M \hookrightarrow L$ extending $\jmath$. Then $r \leq d$, with equality $\iff$ M/K is separable and $\jmath(m_\alpha)$ splits over L for every $\alpha \in M$.*

For the proof, it is helpful to isolate a special case:

I.F.4. LEMMA. *Let $K(\alpha)/K$ be a simple extension of degree d, with L and r as in the Theorem. Then $r \leq d$, with equality $\iff$ α is separable over K and $\jmath(m_\alpha)$ splits over L.*

PROOF. By I.C.14, we get exactly one embedding of $K(\alpha)$ into L extending $\jmath$ for each of the (say) r distinct roots of $\jmath(m_\alpha)$ in L. Clearly $r \leq \deg(\jmath(m_\alpha)) = d$, with equality $\iff$ $\jmath(m_\alpha)$ splits into *distinct* linear factors $\iff$ $\jmath(m_\alpha)$ is separable/$\jmath(K)$ and $\jmath(m_\alpha)$ splits/L. □

PROOF OF I.F.3. We proceed by induction on d:

($\Longleftarrow$): Let $\alpha \in M \setminus K$, and note that $M/K(\alpha)$ is separable (of degree $< d$) by I.F.2. To each of the $[K(\alpha):K]$ distinct embeddings $\sigma\colon K(\alpha) \hookrightarrow L$ produced by the Lemma, we may apply the inductive hypothesis to extend σ in $[M:K(\alpha)]$ distinct ways to $M \hookrightarrow L$, and conclude by the Tower Law . . . provided $\sigma(\mu_\beta)$ splits over L for every β in M (with μ_β its minimal polynomial over $K(\alpha)$). But this is routine: in $K(\alpha)[x]$, μ_β divides the minimal polynomial m_β over K; and $\sigma(m_\beta)$ (hence $\sigma(\mu_\beta)$) splits over L by hypothesis.

($\Longrightarrow$): If for some $\alpha \in M$, $\jmath(m_\alpha)$ doesn't split or α is inseparable over K, then by the Lemma there are fewer than $[K(\alpha):K]$ ways to extend $\jmath$ to $K(\alpha)$. Applying induction to $M/K(\alpha)$ and the Tower Law, there are thus fewer than d ways to extend $\jmath$ to M. □

Normal extensions.

I.F.5. DEFINITION. A field extension L/K is called **normal** if it is algebraic and each irreducible $f \in K[x]$ either splits[18] over L or has no roots in L. Equivalently, L/K is algebraic and, for each $\alpha \in L$, the minimal polynomial $m_\alpha \in K[x]$ splits over L.

In particular, $K(\alpha)/K$ is normal if and only if α is algebraic over K and m_α splits over $K(\alpha)$. Already one sees that this doesn't hold for $K = \mathbb{Q}$ and $\alpha = 2^{\frac{1}{3}}$. Unlike separability, normality will routinely fail

[18]Remember that "splits" always means "into linear factors".

in characteristic zero. Fortunately, as we shall see, there is a "fix" of sorts: if an algebraic extension L/K is non-normal, there will always be a larger extension containing it which *is* normal.

I.F.6. DEFINITION. L/K is a **splitting field extension** for a (possibly infinite) subset $\mathcal{S} \subset K[x]$ if:

(i) every $f \in \mathcal{S}$ splits over L; and
(ii) any proper subfield of L fails to split some $f \in \mathcal{S}$.

I.F.7. PROPOSITION. *L/K is normal $\iff$ L is a splitting field extension for some $\mathcal{S} \subset K[x]$.*

PROOF. ($\underline{\Longrightarrow}$): Set $\mathcal{S}_L := \{m_\alpha \mid \alpha \in L\} \subset K[x]$. Then each $f \in \mathcal{S}_L$ splits over L; and any proper subfield of L omits some $\alpha \in L$ hence doesn't split its m_α.

($\underline{\Longleftarrow}$): Set $R_\mathcal{S} := \{\alpha \in L \mid f(\alpha) = 0 \text{ for some } f \in \mathcal{S}\}$. Then $L = \overline{K(R_\mathcal{S})}$ by I.F.6(ii), hence is algebraic/K by I.A.21. It remains, given $\beta \in L$, to show that $m_\beta \in K[x]$ splits/L. More precisely, we have $\beta \in K(\alpha_1, \ldots, \alpha_n)$ for some $\alpha_i \in R_\mathcal{S}$ (with $f_i(\alpha_i) = 0$ for some $f_i \in \mathcal{S}$); and writing $g := f_1 \cdots f_n$ and $R_g := \{a \in L \mid g(a) = 0\} \subset L$, we note that $\beta \in K(R_g)$. It will suffice to show that m_β splits/$K(R_g)$.

So let M be a splitting field extension for m_β *over* $K(R_g)$, and $\gamma \in M$ a root of m_β. Since the latter is also the minimal polynomial over K for γ, by I.C.16(b) we get τ in

$$\begin{array}{ccccc} & & K(\beta) & \xrightarrow{\text{SFE for } g} & K(R_g) \\ K & \nearrow & \cong \downarrow \tau & & \cong \downarrow \sigma \\ & \searrow & K(\gamma) & \xrightarrow[\tau(g)=g]{\text{SFE for}} & K(R_g, \gamma). \end{array}$$

hence (from I.C.19) σ. So $[K(R_g){:}K(\beta)] = [K(R_g, \gamma){:}K(\gamma)] \implies$

$$\begin{aligned} [K(R_g){:}K] &= [K(R_g){:}K(\beta)][K(\beta){:}K] \\ &= [K(R_g, \gamma){:}K(\gamma)][K(\gamma){:}K] \\ &= [K(R_g, \gamma){:}K] \end{aligned}$$

$\implies K(R_g) = K(R_g, \gamma) \implies \gamma \in K(R_g)$. Conclude that $M = K(R_g)$, i.e. m_β splits over $K(R_g)$ (hence L). □

I.F.8. COROLLARY. *Assume* $[L{:}K] < \infty$. *Then* L/K *is normal* $\iff$ L *is a splitting field for some* $g \in K[x]$.

PROOF. ($\implies$): We may write $L = K\langle a_1, \ldots, a_d\rangle$ (as a vector space). Since L is normal, and each m_{a_i} *has* a root (namely, a_i) in L, each m_{a_i} must split over L. Moreover, $g := \prod_i m_{a_i}$ splits over no proper subfield of L, since that would not contain all the a_i so would not contain all the roots of g.

($\impliedby$): Immediate from I.F.7. □

I.F.9. REMARK. More explicitly, if L/K is finite, then for some $\alpha_1, \ldots, \alpha_k \in L$ we may write $L = K(\alpha_1, \ldots, \alpha_k)$; and (by the same argument as in I.F.8) L/K is normal if and only if it is a splitting field extension for $g := \prod_{i=1}^k m_{\alpha_i}$.

I.F.10. EXAMPLE. As previously mentioned, $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not normal: $m_{\sqrt[3]{2}}(x) = x^3 - 2 \in \mathbb{Q}[x]$ factors into the irreducible factors $x - \sqrt[3]{2}$ and $x^2 + \sqrt[3]{2}x + (\sqrt[3]{2})^2$ in $\mathbb{Q}(\sqrt[3]{2})[x]$, so does *not* split there. How do I know that the quadratic factor is irreducible (and can I see this by pure thought)? Well, by the argument in the proof of I.F.7, you see that any two splitting field extensions have the same degree. We know that $L := \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ is a splitting field for $m_{\sqrt[3]{2}}$ over $\mathbb{Q}$, since it is what you get by adjoining the complex roots; and by the Tower Law, $3 = [\mathbb{Q}(\sqrt[3]{2}){:}\mathbb{Q}]$ and $2 = [\mathbb{Q}(\zeta_3){:}\mathbb{Q}]$ both divide $[L{:}\mathbb{Q}]$, which is therefore 6.

I.F.11. EXAMPLE. The cyclotomic extension[19] $\mathbb{Q}(\zeta_m)/\mathbb{Q}$ is normal for any m. The other roots of the minimal polynomial m_{ζ_m} are powers of ζ_m, since $\zeta_m \mid x^m - 1$ (and all roots of $x^m - 1$ are powers of ζ_m). Clearly, these roots also generate the extension.

[19] As usual, $\zeta_m = e^{2\pi \mathrm{i}/m}$ denotes a primitive root of 1 in $\mathbb{C}$.

While we have seen that intermediate fields in a normal extension L/K need not be normal over K, one can take a different perspective:

I.F.12. COROLLARY. *Given L/K normal, $M \subset L$ a subfield containing K, L/M is normal.*

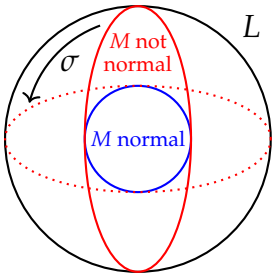
PROOF. By I.F.7 there exists $\mathcal{S} \subset K[x]$ such that L/K is a SFE for $\mathcal{S}$. But then, if we consider $\mathcal{S}$ as a subset of $M[x]$, L/M is evidently a SFE for $\mathcal{S}$. Applying I.F.7 again, we get that L/M is normal. □

We are now ready to consider the relationship between normality and automorphisms, which both suggests the reason for the terminology and provides the first real taste of Galois theory. Write

$$\text{(I.F.13)} \qquad \mathrm{Aut}(L/K) := \{\sigma \in \mathrm{Aut}(L) \big| \sigma|_K = \mathrm{id}_K\}.$$

I.F.14. THEOREM. *Let L/K be finite and normal, and $M \subset L$ a subfield containing K. Then the following are equivalent:*

(i) *M/K is normal;*
(ii) *$\sigma \in \mathrm{Aut}(L/K) \implies \sigma(M) \subseteq M$; and*
(iii) *$\sigma \in \mathrm{Aut}(L/K) \implies \sigma(M) = M$.*



PROOF. (i) $\implies$ (ii): Given $\alpha \in M$, let $m_\alpha \in K[x]$ be the minimal polynomial, and $R_{m_\alpha} \subset L$ its roots. If M/K is normal, we have $R_{m_\alpha} \subset M$; and $m_\alpha(\sigma(\alpha)) = \sigma(m_\alpha(\alpha)) = 0 \implies \sigma(\alpha) \in R_{m_\alpha} \subset M$. So $\sigma(M) \subset M$.

(ii) $\implies$ (iii): This works because L/K is finite and σ is injective, since then $\sigma(M) \subseteq M \implies [M{:}K] \geq [\sigma(M){:}K] = [M{:}K]$ forces $\sigma(M) = M$. (Think K-vector spaces.)

(iii) $\implies$ (i): Given $\alpha \in M$, we must show that $m_\alpha \in K[x]$ splits over M, or equivalently that its roots $R_{m_\alpha} \subset L$ lie in M. Let $\beta \in R_{m_\alpha}$.

By I.F.8, L is a splitting field for some $g \in K[x]$ — over K, hence also over $K(\alpha)$ and $K(\beta)$. Moreover, I.C.16(b) gives an isomorphism $\jmath\colon K(\alpha) \to K(\beta)$ over K sending $\alpha \mapsto \beta$. Since $\jmath(g) = g$, by I.C.19

there is a $\sigma \in \mathrm{Aut}(L)$ extending $\jmath$ hence fixing K; and by assumption of (iii), we have $\sigma(M) = M$. So we have $\beta = \jmath(\alpha) = \sigma(\alpha) \in M$. □

Normal closures.

I.F.15. DEFINITION. A **normal closure** L^c for an algebraic extension L/K is an extension L^c/L such that

(i) L^c/K is normal; and
(ii) $L^c \supset M \supset L$ and M/K normal $\implies M = L^c$.

I.F.16. COROLLARY. *If L/K is finite, then it has a finite normal closure ($\implies L^c/K$ finite).*

PROOF. As in the proof of I.F.8, write $L = K\langle a_1, \ldots, a_d\rangle$, $g = \prod m_{a_i}$, and consider F/L a SPE for g. Then F/K is a SFE for g, hence normal. Moreover, if $F \supset M \supset L$ with M/K normal, then g splits$/M$, hence $M = F$. □

I.F.17. EXAMPLE. $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ is a normal closure for $\mathbb{Q}(\sqrt[3]{2})$.

I.F.18. PROPOSITION. *Suppose $L = K(\alpha_1, \ldots, \alpha_r)/K$ is finite, with α_i separable over $K_{i-1} := K(\alpha_1, \ldots, \alpha_{i-1})$ for each i. Then L/K is separable.*

PROOF. Let L^c/L be a normal closure.

At each stage, by I.F.4 there exist $[K_i{:}K_{i-1}]$ embeddings $K_i \hookrightarrow L^c$ extending a given embedding $K_{i-1} \hookrightarrow L^c$. (Note that the minimal polynomial of α_i over K_{i-1} divides its minimal polynomial over K, hence splits over L^c because the latter does. So I.F.4 does indeed apply.) By the Tower Law, this amounts to $[L{:}K]$ distinct embeddings of L into L^c. Applying I.F.3 (with M, L there replaced by L, L^c here), we find that L/K is separable. □

I.F.19. COROLLARY. *A finite extension L/K is separable in any of the following cases:*
(a) *$L = K(\alpha_1, \ldots, \alpha_r)$ and each α_i is separable$/K$.*
(b) *$f \in K[x]$ is separable$/K$ and L/K is a SFE for f.*
(c) *$L \supset M \supset K$ with L/M and M/K separable.*

PROOF. For (a), note that (with notation from I.F.18) each α_i being separable over K makes it also separable over K_{i-1} (by the proof of I.F.2); now apply I.F.18. For (b), apply (a) to roots of f in L. For (c), write $L = M(\{\beta_j\})$ and $M = K(\{\alpha_i\})$; apply I.F.2 to split L/M and M/K into successive simple separable extensions; then apply I.F.18 to the whole tower. □

Galois extensions.

I.F.20. DEFINITION. A **Galois extension** is a finite, normal, separable extension.

I.F.21. THEOREM. *Let L/K be finite. Then $|\mathrm{Aut}(L/K)| \leq [L{:}K]$, with equality $\iff$ L/K is Galois.*

PROOF. Apply I.F.3 with "M, L" both L. The "embeddings extending $\jmath$" become (by I.A.23) automorphisms fixing K, and I.F.3 says precisely that there are $[L{:}K]$ of these iff L/K is separable and normal. □

I.F.22. COROLLARY. *An extension L/K is Galois if and only if it is a splitting field extension for a separable polynomial $f \in K[x]$.*

PROOF. If L/K is Galois, then it is the SFE for some $f \in K[x]$ by I.F.8; since this f is a product of minimal polynomials of its roots, it is separable by I.F.1.

For the other direction, given $f \in K[x]$, the SFE is finite and normal (cf. I.F.8), and separable if f is (cf. I.F.19(b)). □

I.F.23. DEFINITION. If L/K is Galois, $\mathrm{Aut}(L/K)$ is called the **Galois group**;[20] it is standard to write $\mathbf{Gal(}\boldsymbol{L/K}\mathbf{)}$ for $\mathrm{Aut}(L/K)$ in this case.

[20]Note that [**Jacobson**] actually calls $\mathrm{Aut}(L/K)$ the "Galois group" (and writes $\mathrm{Gal}(L/K)$) unconditionally. One should be a bit wary here, as some authors (at least, for L/K separable) define the Galois group of L/K to be $\mathrm{Aut}(L^c/K)$. I am happier to drop the finiteness condition on L/K in favor of algebraicity, so that we can speak of (for example) $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, which is very much standard usage. But definitions are definitions and we shall stick with ours.