## II.E. Hermitian forms and unitary groups

Assuming as usual that $\mathrm{char}(\mathbb{F}) \neq 2$, let $\mathbb{E}/\mathbb{F}$ be a field extension of degree 2, and $\rho \in \mathrm{Aut}(\mathbb{E}/\mathbb{F})$ the generator (with $\rho^2 = \mathrm{id}_{\mathbb{E}}$); we shall write $\bar{\alpha}$ for $\rho(\alpha)$. Let $V$ be a vector space of dimension $n$ over $\mathbb{E}$; considering this as a vector space of dimension $2n$ over $\mathbb{F}$ defines the so-called *Weil-restriction* $W := \mathrm{Res}_{\mathbb{E}/\mathbb{F}} V$. Remember that we can also think of $\mathbb{E}$ as a 2-dimensional vector space over $\mathbb{F}$.

II.E.1. DEFINITION. (i) A matrix $\boldsymbol{h} = (h_{ij})$ is **Hermitian** if $h_{ji} = \bar{h}_{ij}$ $(\forall i, j)$, i.e. if $\boldsymbol{h}^* := {}^t\bar{\boldsymbol{h}} = \boldsymbol{h}$. It is **skew-Hermitian** if $\boldsymbol{h}^* = -\boldsymbol{h}$.

(ii) A **Hermitian form** on $V$ is an element

$$H \in \mathrm{Hom}_{\mathbb{F}}(W \otimes W, \mathbb{E})$$

(or, if you prefer, a map from $V \times V \to \mathbb{E}$) satisfying

(a) $H(\alpha x + \beta y, v) = \alpha H(x, v) + \beta H(y, v)$,
(b) $H(v, \alpha x + \beta y) = \bar{\alpha} H(v, x) + \bar{\beta} H(v, y)$, and
(c) $\overline{H(u, v)} = H(v, u)$.

(Actually, (a) and (c) imply (b).) The pair $(V, H)$ is called a **Hermitian vector space**.

Given a basis $e = \{e_1, \ldots, e_n\}$ of $V$ over $\mathbb{E}$, and writing $[H]_e =: \boldsymbol{h}$, computing $H(\sum_i u_i e_i, \sum_j v_j e_j)$ immediately yields

$$H(u, v) = {}^t[u]_e \boldsymbol{h} \overline{[v]_e}. \tag{II.E.2}$$

Clearly, (c) is equivalent to

$${}^t\overline{[u]_e}\bar{\boldsymbol{h}}[v]_e = {}^t[v]_e \boldsymbol{h} \overline{[u]_e} \left( = {}^t({}^t[v]_e \boldsymbol{h} \overline{[u]_e}) = {}^t\overline{[u]_e}{}^t\boldsymbol{h}[v]_e \right) \quad (\forall u, v \in V)$$

and thus to $\boldsymbol{h}$ being a Hermitian matrix.

As was the case for symmetric and alternating bilinear forms, there is a particularly nice basis that puts this matrix in a "standard form":

II.E.3. THEOREM. *Let $(V, H)$ be a Hermitian vector space over $\mathbb{E}$. Then $V$ has a basis $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_n\}$ for which $H(\varepsilon_i, \varepsilon_j) = b_j \delta_{ij}$, with $b_j \in \mathbb{F}^*$ for $1 \leq j \leq r$ and $b_j = 0$ for $r < j \leq n$.*

PROOF. Suppose that $H(u,u) = 0$ for all $u \in V$. If $H$ is not identically zero, then there exist $u, v$ with $H(u,v) = 1$. Then $0 = H(u+v,u+v) = H(u,v) + H(v,u) = H(u,v) + \overline{H(u,v)} = 2$, a contradiction. So, assuming $H$ is not trivial, there is an $\varepsilon_1 \in V$ with $b_1 := H(\varepsilon_1, \varepsilon_1) \neq 0$. By Hermitian symmetry (property (c) in II.E.1(ii)), $b_1 = \bar{b}_1$ hence $b_1 \in \mathbb{F}^*$.

Inductively suppose we have $\varepsilon_1, \ldots, \varepsilon_k$ with $H(\varepsilon_i, \varepsilon_j) = b_i \delta_{ij}$, and let $U_k$ be their span. We have $V = U_k \oplus U_k^{\perp}$ since we can write any $x \in V$ as $y + (x - y)$ where $y := \sum_{j=1}^{k} b_j^{-1} H(x, \varepsilon_j)\varepsilon_j$. Apply the last paragraph to $H|_{U_k^{\perp}}$ to complete the inductive step. □

Since a change of basis (with matrix $S$) transforms $\boldsymbol{h} \mapsto {}^t S \boldsymbol{h} \bar{S}$, we can evidently multiply each $b_j$ by anything in $N_{\mathbb{E}/\mathbb{F}}(\mathbb{E}^*)$ (i.e. "absolute squares" $\alpha\bar{\alpha}$) for free. If $\mathbb{E} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$, the same argument as in the proof of Sylvester's theorem shows, conversely, that we cannot change the number $p$ and $q$ of positive resp. negative $b_j$'s. So the Hermitian forms in this case are completely classified by the pair $(p,q)$, which is again called the **signature**.

**The associated bilinear forms.**

As Definition II.E.1 points out, while $H$ is only $\mathbb{E}$-*sesquilinear* (this is (a) and (b)), it is $\mathbb{F}$-bilinear. That means its "real" and "imaginary" parts (so to speak) should yield $\mathbb{F}$-bilinear forms on $W$, giving us access to results we have already proved in this case. Indeed, writing $\mathbb{E} = \mathbb{F}(\sqrt{f})$ (as we can do when $\mathrm{char}(\mathbb{F}) \neq 2$), define $B', B'' \in \mathrm{Hom}(W \otimes W, \mathbb{F})$ by

$$H(w,u) =: B'(w,u) - \sqrt{f} B''(w,u). \tag{II.E.4}$$

Equivalently, we can set

$$\begin{cases} B'(w,u) := \frac{1}{2}\left(H(w,u) + H(u,w)\right) \quad \text{and} \\ B''(w,u) := \frac{-1}{2\sqrt{f}}\left(H(w,u) - H(u,w)\right). \end{cases} \tag{II.E.5}$$

which makes it clear that $B'$ is symmetric and $B''$ alternating.

Now consider the multiplication-by-$\sqrt{f}$ map $\mu \in \mathrm{End}_{\mathbb{F}}(W)$. This has minimal polynomial $x^2 - f$, hence "diagonalizes over $\mathbb{E}$ with $\rho$-conjugate eigenspaces and eigenvalues":

$$W \otimes_{\mathbb{F}} \mathbb{E} \cong E_\mu(\sqrt{f}) \oplus E_\mu(-\sqrt{f}) \cong V \oplus \bar{V}, \tag{II.E.6}$$

where "$\bar{V}$" is a copy of $V$ on which $\mathbb{E}$ acts through $\rho$. Sending $\xi \mapsto (\xi, \bar{\xi})$ (acting on RHS(II.E.6)) thus yields an embedding

$$\mathrm{End}_{\mathbb{E}}(V) \hookrightarrow \mathrm{End}_{\mathbb{F}}(W)$$

with image contained in $\mathrm{End}_{\mathbb{F}}(W)^{\mu}$, i.e. the endomorphisms commuting with $\mu$. In fact, this *is the image*, since any $\mathbb{F}$-linear endomorphism commuting with $\mu$ simultaneously diagonalizes with $\mu$ hence sends (over $\mathbb{E}$) $V \to V$ and $\bar{V} \to \bar{V}$, and takes the form $(\xi, \bar{\xi})$.

II.E.7. REMARK. As an aside here, it is useful to keep in mind that if you represent $\mathbb{C}$ as an $\mathbb{R}$-vector-space with basis $\{1, \mathbf{i}\}$, then multiplication by $\alpha = a + b\mathbf{i}$ is represented by the *matrix* $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. This matrix commutes with $[\mu] = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (multiplication by $\sqrt{f} = \mathbf{i}$) and diagonalizes *over* $\mathbb{C}$ as $\mathrm{diag}(a + b\mathbf{i}, a - b\mathbf{i})$.

That is, you begin with $\mathbb{C}$ as a 1-dimensional vector space over itself, Weil-restrict to $W := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{C} = \mathbb{R}\langle 1, \mathbf{i}\rangle$, then extend coefficients to $\mathbb{C}$ via $W \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$. The two copies of $\mathbb{C}$ are spanned by $\frac{1}{2}\left(\begin{smallmatrix} 1 \\ -\mathbf{i} \end{smallmatrix}\right)$ and $\frac{1}{2}\left(\begin{smallmatrix} 1 \\ \mathbf{i} \end{smallmatrix}\right)$, and multiplication by $\alpha$ acts through $\alpha$ on the first copy and $\bar{\alpha}$ on the second.

Returning to our bilinear forms $B', B''$ over $\mathbb{F}$, for arbitrary $w, u \in W$ ($= V$ as a set) we compute

$$\begin{aligned} B''(w, \mu u) &= \tfrac{-1}{2\sqrt{f}} \left( H(w, \mu u) - H(\mu u, w) \right) \\ &= \tfrac{-1}{2\sqrt{f}} \left( -\sqrt{f} H(w, u) - \sqrt{f} H(u, w) \right) \\ &= \tfrac{1}{2} \left( H(w, u) + H(u, w) \right) \; = \; B'(w, u), \end{aligned}$$

which also implies $B''(w,u) = \frac{1}{f}B'(w,\mu u)$. One consequence is that we can write $H$ in terms of just $B'$ *or* $B''$:
(II.E.8)
$$H(w,u) = B'(w,u) - \frac{1}{\sqrt{f}}B'(w,\mu u) = -\sqrt{f}B''(w,u) + B''(w,\mu u).$$

Another is that, given a transformation $T \in \mathrm{End}_{\mathbb{F}}(W)^{\mu}$, we have

$$\begin{aligned} T \in \mathrm{Aut}(W,B') &\iff B'(Tw,Tu) = B'(w,u) \ \ (\forall w,u \in W) \\ &\iff B''(Tw,\mu Tu) = B''(w,\mu u) \ \ (\forall w,u \in W) \\ &\underset{z=\mu u}{\iff} B''(Tw,Tz) = B''(w,z) \ \ (\forall w,z \in W) \\ &\iff T \in \mathrm{Aut}(W,B''). \end{aligned}$$

The upshot is that we needn't worry about both $B'$ and $B''$, so we will make a choice and drop $B'$.

II.E.9. REMARK. There is a different perspective on how $H$ and $B''$ are related. Using the identification of $V$ with $W$ (as sets), we map $V = W \hookrightarrow W \otimes_{\mathbb{F}} \mathbb{E} \cong E_{\mu}(\sqrt{f}) \oplus E_{\mu}(-\sqrt{f})$, denoting the image of $v$ by $(\imath(v), \bar{\imath}(\bar{v}))$. (In the setting of II.E.7, the effect is to send $\alpha = a + b\mathbf{i} \mapsto \frac{\alpha}{2}\binom{1}{-\mathbf{i}} + \frac{\bar{\alpha}}{2}\binom{1}{\mathbf{i}}$.) This *defines* isomorphisms $\imath\colon V \to E_{\mu}(\sqrt{f})$ and $\bar{\imath}\colon \bar{V} \to E_{\mu}(-\sqrt{f})$, explaining the RHS of (II.E.5) a little better. That is, we can write $v = \imath(v) + \bar{\imath}(\bar{v})$ in $W \otimes_{\mathbb{F}} \mathbb{E}$.

Writing $B''_{\mathbb{E}}$ for the $\mathbb{E}$-linear extension of $B''$ to $W \otimes_{\mathbb{F}} \mathbb{E}$, I claim that $B''_{\mathbb{E}}(\imath(v_1), \imath(v_2)) = 0 = B''_{\mathbb{E}}(\bar{\imath}(\bar{v}_1), \bar{\imath}(\bar{v}_2))$. To see this, observe that[12]

$$B''(\mu w, \mu z) = B'(\mu w, z) = B'(z, \mu w) = fB''(z,w) = -fB''(w,z)$$

hence [by $\mathbb{E}$-linear extension]

$$\begin{aligned} -fB''(\imath(v_1), \imath(v_2)) &= B''(\mu(\imath(v_1)), \mu(\imath(v_2))) \\ &= B''(\sqrt{f}\imath(v_1)), \sqrt{f}\imath(v_2)) = fB''(\imath(v_1), \imath(v_2)). \end{aligned}$$

The upshot that $E_{\mu}(\sqrt{f})$ and $E_{\mu}(-\sqrt{f})$ are isotropic for $B''_{\mathbb{E}}$, which only pairs them with each other.

[12] Alternatively here, you could just take the "imaginary" part of $H(\mu v_1, \mu v_2) = H(\sqrt{f}v_1, \sqrt{f}v_2) = -fH(v_1, v_2)$ in (II.E.4).

Using this, (II.E.8), and anti-symmetry of $B''_{\mathbb{E}}$, we calculate

$$\begin{aligned} H(v_1, v_2) &= H\left(\imath(v_1) + \bar{\imath}(\bar{v}_1), \imath(v_2) + \bar{\imath}(\bar{v}_2)\right) \\ &= -\sqrt{f} B''_{\mathbb{E}}\left(\imath(v_1) + \bar{\imath}(\bar{v}_1), \imath(v_2) + \bar{\imath}(\bar{v}_2)\right) \\ &\qquad + B''_{\mathbb{E}}(\imath(v_1) + \bar{\imath}(\bar{v}_1), \sqrt{f}\imath(v_2) - \sqrt{f}\bar{\imath}(\bar{v}_2)) \\ &= -2\sqrt{f} B''_{\mathbb{E}}\left(\imath(v_1), \bar{\imath}(\bar{v}_2)\right). \end{aligned}$$

That is, we can view the Hermitian form on $v_1, v_2$ in terms of the ($\mathbb{E}$-linearly extended) *symplectic* form applied to $v_1 \in V$ and $v_2 \in \bar{V}$ in RHS(II.E.6).

**Unitary groups.**

Henceforth we assume that our Hermitian form $H$ is nondegenerate, which makes $B''$ (and $B'$) nondegenerate by (II.E.8). However, I will write $\mathrm{Sp}(W, B'')$ rather than $\mathrm{Sp}_{2n}(\mathbb{F})$ for its isometries, since we have to worry about how this particular copy of the symplectic group is situated relative to the action of $\mu$.

II.E.10. DEFINITION. (i) The isometry group

$$U(V, H) := \{T \in \mathrm{Aut}_{\mathbb{E}}(V) \mid H(Tu, Tv) = H(u, v)\ (\forall u, v \in V)\}$$

is called a **unitary group**.

(ii) When $\mathbb{E}/\mathbb{F}$ is $\mathbb{C}/\mathbb{R}$, the isomorphism class of this group is determined by the signature $(p, q)$ of $H$, and we write $U(V, H) \cong U(p, q)$.

Recall from §II.B that the sesquilinearity of $H$ means that $U(V, H)$ is not a linear algebraic group over $\mathbb{E}$, but only over $\mathbb{F}$. As a final result, we can use our work above to see this more explicitly.

II.E.11. PROPOSITION. *We have*

$$U(V, H) \cong \mathrm{Sp}(W, B'') \cap \mathrm{Res}_{\mathbb{E}/\mathbb{F}}(\mathrm{Aut}(V)) = \mathrm{Sp}(W, B'')^{\mu}.$$

PROOF. We prove that the end terms are isomorphic. See the remark below for the middle term.

Given $T \in U(V, H)$, $T$ is $\mathbb{E}$-linear so commutes with multiplication by $\sqrt{f}$ (i.e. $\mu$). Moreover, $H(Tu, Tv) = H(u, v)$ becomes by

(II.E.5) $B''(Tu, Tv) = B''(u, v)$ (where we can of course interpret $T$ as an $\mathbb{F}$-linear automorphism of $W$). Hence $T \in \mathrm{Sp}(W, B'')$.

Conversely, if $T \in \mathrm{Sp}(W, B'')$ ($\implies T$ is $\mathbb{F}$-linear) and commutes with $\mu$, then it is $\mathbb{E}$-linear, so belongs to $\mathrm{Aut}_{\mathbb{E}}(V)$. Then one applies (II.E.8) to get from $B''(Tu, Tv) = B''(u, v)$ (and $T\mu = \mu T$) to $H(Tu, Tv) = H(u, v)$. □

II.E.12. REMARK. To understand the middle term in II.E.11, the main point is that the Weil restriction $\mathrm{Res}_{\mathbb{E}/\mathbb{F}}(\mathrm{Aut}_{\mathbb{E}}(V))$ is precisely $\mathrm{Aut}_{\mathbb{F}}(W)^{\mu}$, i.e. automorphisms of $W$ commuting with $\mu$. As with vector spaces, the meaning of $\mathrm{Res}_{\mathbb{E}/\mathbb{F}}$ is to realize a linear algebraic group over $\mathbb{E}$ as one over $\mathbb{F}$. The proof of this is little more than "$\mathbb{F}$-linear plus commutes with $\mu$" = "$\mathbb{E}$-linear".

But it is useful to think about what this means explicitly: given a basis $\{e_1, \ldots, e_n\}$ of $V/\mathbb{E}$, $\{e_1, \sqrt{f}e_1, \ldots, e_n, \sqrt{f}e_n\}$ gives a corresponding basis of $W/\mathbb{F}$. On the level of matrices with respect to these bases, $\mathrm{Res}_{\mathbb{E}/\mathbb{F}}$ replaces each entry $\alpha = a + b\sqrt{f}$ of an $n \times n$ matrix over $\mathbb{E}$ by the $2 \times 2$ block $\begin{pmatrix} a & fb \\ b & a \end{pmatrix}$, thus producing a $2n \times 2n$ matrix over $\mathbb{F}$. One easily checks that such matrices are exactly the $\mu$-invariant ones, where $\mu$ is the block-diagonal matrix $\begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix}$; this is left to you.