III.B. The Artin-Wedderburn theorem

We now turn to the classification of semisimple rings. The main result will say that all semisimple rings are products of matrix algebras over division rings (like fields, quaternions, etc.). We will need several results on endomorphism rings of R-modules before we can get started with the proof.

III.B.1. PROPOSITION. (i) $R^{\text{op}} \cong \text{End}_R({}_R R)$ *and* $R \cong \text{End}_R(R_R)$.
(ii) *For any* R*-module* M, $M_n(\text{End}_R(M)) \cong \text{End}_R(M^{\oplus n})$.

PROOF. (i) This is the $m = 1$ case of [**Algebra I**, IV.B.21], but we can easily recap the proof: define maps

$$\ell \colon R \to \text{End}_R(R_R) \quad \text{and} \quad \mathfrak{r} \colon R^{\text{op}} \to \text{End}_R({}_R R)$$

(sending $r \mapsto \ell_r$ resp. $\mathfrak{r}_r$) by $\ell_r(a) := ra$ and $\mathfrak{r}_r(a) := ar$. One easily checks these are isomorphisms. We do it for $\mathfrak{r}$:

- [homomorphism] $\mathfrak{r}_{r \cdot^{\text{op}} s}(a) = \mathfrak{r}_{sr}(a) = asr = \mathfrak{r}_r(as) = \mathfrak{r}_r(\mathfrak{r}_s(a))$ shows that $r \cdot^{\text{op}} s \mapsto \mathfrak{r}_r \mathfrak{r}_s$.
- [injectivity] $\mathfrak{r}_r = 0 \implies 0 = \mathfrak{r}_r(1) = 1r = r$.
- [surjectivity] $\varphi \in \text{End}_R({}_R R) \implies \mathfrak{r}_{\varphi(1)}(a) = a\varphi(1) = \varphi(a1) = \varphi(a) \implies \mathfrak{r}_{\varphi(1)} = \varphi$.

(ii) This works for both left- and right-R-modules; we will focus on the left ones. Writing $\imath_k \colon M \hookrightarrow M^{\oplus n}$ for the inclusion of the k^{th} direct summand, denote elements of $M^{\oplus n}$ by $\mu = \sum_j \imath_j(\mu_j) = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$. Define a map

$$\begin{aligned} \theta \colon M_n(\text{End}_R(M)) &\to \text{End}_R(M^{\oplus n}) \\ \varepsilon = (\varepsilon_{ij}) &\mapsto \theta_\varepsilon \end{aligned}$$

by $\theta_\varepsilon(\mu) := \sum_{i,j} \imath_i(\varepsilon_{ij}(\mu_j))$. That is, $\theta_\varepsilon \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = \begin{pmatrix} \sum_j \varepsilon_{1j}(\mu_j) \\ \vdots \\ \sum_j \varepsilon_{nj}(\mu_j) \end{pmatrix}$ looks exactly like matrix-vector multiplication, so is automatically a ring homomorphism provided $\theta_\varepsilon \in \text{End}_R(M^{\oplus n})$. The latter is immediate since each of the $\imath_i$ and ε_{ij} are left-R-module homomorphisms.

For injectivity, $0 = \theta_\varepsilon \implies 0 = \theta_\varepsilon(\imath_k(m)) = \sum_{i,j} \imath_i(\varepsilon_{ij}(m\delta_{jk})) = \sum_i \imath_i(\varepsilon_{ik}(m))$ for each k and every $m \in M \implies \varepsilon_{ik}(m) = 0\ (\forall i, k, m) \implies \varepsilon_{ij} = 0\ (\forall i, j) \implies \varepsilon = 0$.

For surjectivity, suppose $\Theta \in \mathrm{End}_R(M^{\oplus n})$. Writing $\rho_i \colon M^{\oplus n} \twoheadrightarrow M$ for the projection onto the i^{th} summand, define $\varepsilon_{ij} \in \mathrm{End}_R(M)$ by $\varepsilon_{ij}(m) := \rho_i(\Theta(\imath_j(m)))$. Then $\theta_\varepsilon(\mu) = \sum_{i,j} \imath_i(\rho_i(\Theta(\imath_j(\mu_j)))) = \sum_j \Theta(\imath_j(\mu_j)) = \Theta(\mu)$ for any $\mu \in M^{\oplus n}$ hence $\theta_\varepsilon = \Theta$. □

Next, we need a more detailed version of Schur's Lemma from [**Algebra I**, IV.B.30-31]. This requires a "prelemma" of sorts which is important in its own right:

III.B.2. LEMMA. *Let D be a division ring. Up to isomorphism, $M_n(D)$ has one* (*left*) *simple module V, namely the one given by its action by left multiplication on "column vectors" in $D^{\oplus n}$.*

PROOF. First, it is clear that "$D^{\oplus n}$" *is* a simple module, since $M_n(D)$ acts transitively on its nonzero elements. Moreover, if we consider $M_n(D)$ as a (left) module over itself, it decomposes into a direct sum of n copies of this module, each given by matrices with nonzero entries in only one column. If we call these "column vector submodules" $\{C_i\}_{i=1}^n$, then

$$\{0\} \subset C_1 \subset C_1 \oplus C_2 \subset \cdots \subset C_1 \oplus \cdots \oplus C_n = {}_{M_n(D)}M_n(D)$$

gives a composition series, each of whose simple graded pieces is our "$D^{\oplus n}$".

Now let V be any simple $M_n(D)$-module. Then $V = M_n(D)v$ is cyclic. The map $\theta \colon M_n(D) \twoheadrightarrow M_n(D)v$ has a maximal (left) ideal as its kernel. (If it wasn't maximal, then by 1st isomorphism for modules, $V \cong M_n(D)/\ker(\theta)$ wouldn't be simple.) By the same argument as in the proof of Jordan-Hölder, we know that $\ker(\theta)$ has a composition series. (Or use your HW exercise: a submodule of a Noetherian and Artinian module retains those properties.) This makes $\ker(\theta)$ the first submodule in a CS for ${}_{M_n(D)}M_n(D)$ itself. By Jordan-Hölder itself, together with the last paragraph, we conclude that V is isomorphic to our "$D^{\oplus n}$" module. □

III.B.3. SCHUR'S LEMMA. *Let $V \ncong W$ be simple R-modules.*

(i) $\mathrm{Hom}_R(V,W) = \{0\}$.

(ii) $\mathrm{Hom}_R(V,V)$ $(\mathrm{End}_R(V))$ *is a division ring.*

(iii) *If $R \cong M_n(D)$, where D is a division ring, then $\mathrm{End}_R(V) \cong D^{op}$. In particular, if R is a division ring, then $\mathrm{End}_R(V) \cong R^{op}$.*

PROOF. We recap the previous proof. Given any R-module homomorphism $\theta\colon V \to V'$, $\ker(\theta)$ is a submodule of V. Since V is simple, this is V or zero; so θ is zero or injective. If V' is also simple then $\theta(V)$ must be zero or all of V'. Hence θ is zero or an isomorphism, which gives (i) if $V' = W$. It gives (ii) if $V' = V$, because then every nonzero $\theta \in \mathrm{End}_R(V)$ has an inverse.

Turning to (iii), we have $V \cong D^{\oplus n}$ from III.B.2. Define a map $\mathfrak{r}\colon D^{\mathrm{op}} \to \mathrm{End}_R(V)$ by $d \mapsto \mathfrak{r}_d$, i.e. right scalar-multiplication on "vectors" in $D^{\oplus n}$. We have $\mathfrak{r}_d \in \mathrm{End}_R(V)$ since $\mathfrak{r}_d(rv) = rvd = r\mathfrak{r}_d(v)$, the point being that "right and left actions don't interfere with each other". Clearly $\mathfrak{r}$ is an injective ring homomorphism.

Now given any $\varepsilon \in \mathrm{End}_R(V)$, we need to show that $\varepsilon = \mathfrak{r}_d$ for some $d \in D$. Let $\mathbf{e}_i$ and ${}^t\mathbf{e}_i$ denote the standard basis column and row vectors; thinking of $\mathbf{e}_i \in D^{\oplus n} = V$, we write $\varepsilon(\mathbf{e}_1) =: \sum_i \varepsilon_i \mathbf{e}_i$ (with $\varepsilon_i \in D$). Note that the column-row product $[v\,{}^t\mathbf{e}_1]$ is a matrix in $M_n(D) = R$, and the row-column product ${}^t\mathbf{e}_1\mathbf{e}_1 = 1$. So we have

$$\begin{aligned} \varepsilon(v) &= \varepsilon(v\,{}^t\mathbf{e}_1\mathbf{e}_1) = \varepsilon([v\,{}^t\mathbf{e}_1].\mathbf{e}_1) = [v\,{}^t\mathbf{e}_1].\varepsilon(\mathbf{e}_1) \\ &= [v\,{}^t\mathbf{e}_1].\textstyle\sum_i \varepsilon_i \mathbf{e}_i = v \sum_i \varepsilon_i\,{}^t\mathbf{e}_1\mathbf{e}_i = v\sum_i \varepsilon_i \delta_{1i} \\ &= v\varepsilon_1 = \mathfrak{r}_{\varepsilon_1} v. \end{aligned}$$

Since this computation is independent of $v \in V$, it gives $\varepsilon = \mathfrak{r}_{\varepsilon_1}$. □

Note that in the proof, $\mathfrak{r}$ identifies V as a free right D-module (cf. [**Algebra I**, IV.A.11]). Of course, it is also a free left D-module via left-multiplication; this just doesn't commute with the left R-module structure. We also remark (for use below) that taking the transpose of matrices induces an isomorphism $M_n(D^{\mathrm{op}}) \cong M_n(D)^{\mathrm{op}}$.

Now for the main result: you may be surprised by how short the proof is.

III.B.4. THEOREM (Wedderburn, 1907; Artin, 1927).

(a) *Any semisimple ring R can be expressed uniquely as a product of matrix rings over division rings,*

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r). \tag{III.B.5}$$

Conversely, all such products are semisimple.

(b) *There are exactly r non-isomorphic* (*left*) *simple modules V_i over R, namely the "column modules" of each matrix ring above. For $i \neq j$, even if $n_i = n_j$ and $D_i \cong D_j$, we have $V_i \not\cong V_j$, for it is different factors in the product* (III.B.5) *that operate nontrivially on them.*

(c) *As a left module over itself, we have ${}_R R \cong \oplus_{i=1}^r V_i^{\oplus n_i}$, where $n_i =$* $\dim_{D_i}(V_i)$ (*in the sense that $V_i = D_i^{\oplus n_i}$ as D_i-module*).

PROOF. We are assuming that R is (left) semisimple. This means that all its (left) modules decompose into irreducibles; in particular, we can write

$${}_R R \cong \oplus_{i=1}^r V_i^{\oplus n_i}, \tag{III.B.6}$$

with V_i pairwise nonisomorphic simple left R-modules. This sum is finite by III.A.9. (We are not yet assuming that the V_i are the modules described in (b).)

I claim that RHS(III.B.6) contains every simple left R-module. Indeed, if V is any such, we can argue as in the proof of III.B.2 that it is the top graded piece of a composition series for ${}_R R$. But (III.B.6) also provides an obvious CS for ${}_R R$, with the V_i as graded pieces. By Jordan-Hölder, V is isomorphic to one of the V_i.

Now set $D_i := (\mathrm{End}_R V_i)^{\mathrm{op}}$ (division rings by III.B.3(ii)), and compute

$$\begin{aligned} R^{\mathrm{op}} &\underset{\text{III.B.1(i)}}{\cong} \mathrm{End}_R({}_R R) \underset{\text{(III.B.6)}}{\cong} \mathrm{End}_R(\oplus_{i=1}^r V_i^{\oplus n_i}) \\ &\underset{\text{III.B.3(i)}}{\cong} \times_{i=1}^r \mathrm{End}_R(V_i^{\oplus n_i}) \underset{\text{III.B.1(ii)}}{\cong} \times_{i=1}^r M_{n_i}(\mathrm{End}_R(V_i)) \\ &\cong \quad \times_{i=1}^r M_{n_i}(D_i^{\mathrm{op}}). \end{aligned}$$

By previous remarks, applying "op" to both sides gives us that $R \cong \times_{i=1}^{r} M_{n_i}(D_i^{\mathrm{op}})^{\mathrm{op}} \cong \times_{i=1}^{r} M_{n_1}(D_i)$. The $M_{n_i}(D_i)$-modules provided by III.B.2 become, by composing with the projections $R \twoheadrightarrow M_{n_i}(D_i)$, r pairwise nonisomorphic R-modules. These must be the V_i, proving (b) and (c).

The converse statement in (a) follows, via III.A.7, from the decompostion of the matrix ring product into column-vector modules. For the uniqueness statement, observe that if $M \cong \times_{j=1}^{r'} M_{n'_j}(D'_j)$ is another product decomposition, then the corresponding *module* decomposition ${}_R R \cong \oplus_{j=1}^{r'} V_j'^{\oplus n'_j}$ (into column modules) has $D'_j \cong \mathrm{End}_R(V'_j)^{\mathrm{op}}$ by III.B.3(iii). Since these are composition factors, we must have $r = r'$, $n_i = n'_i$ and $V_i \cong V'_i$ (up to permutation) by Jordan-Hölder, and thus $D_i \cong D'_i$ as well. □

Of course all of this works for "right semisimple" rings, with slight modifications to the proofs. So, as a byproduct of III.B.4, we see that left and right semisimple are the same. We also get a classification of (left or right) simple rings:

III.B.7. COROLLARY. *Any simple ring R can be expressed uniquely as a matrix ring over a division ring, $R \cong M_n(D)$. Conversely, all such matrix rings are simple.*

PROOF. If R is simple, it is semisimple, hence $\cong M_{n_1}(D) \times \cdots \times M_{n_r}(D_r)$ by III.B.4. Clearly we must also have $r = 1$ since otherwise the individual factors $0 \times \cdots \times M_{n_i}(D_i) \times \cdots \times 0$ are nontrivial, proper, two-sided ideals in R.

On the other hand, any $M_n(D)$ is simple, since the approach indicated for $M_n(\mathbb{C})$ in III.A.6 shows that any nonzero *two-sided* ideal is the whole ring. □