III.D. Characters of representations

Let G be a finite group, and

$$\text{(III.D.1)} \qquad {}_{\mathbb{C}[G]}\mathbb{C}[G] = \oplus_{\lambda=1}^{r} V_{\lambda}^{\oplus n_\lambda}, \qquad n_\lambda = \dim_{\mathbb{C}}(V_\lambda)$$

the decomposition of the regular representation of G into irreducible representations over $\mathbb{C}$ (a.k.a. simple left $\mathbb{C}[G]$-modules) guaranteed by III.C.5. That is, we now think of the V_λ's as $\mathbb{C}$-vector spaces on which G acts through homomorphisms

$$\text{(III.D.2)} \qquad \pi_\lambda \colon G \to \mathrm{Aut}_{\mathbb{C}}(V_\lambda),$$

in such a way that the image stabilizes no proper nonzero subspace of V_λ. It is remarkable that all of the possible such "irreps" are already present inside $\mathbb{C}[G]$.

Bases for the regular representation.

Let V_{reg} denote $\mathbb{C}[G]$ considered as a vector space, and write $g.v := \pi_{\mathrm{reg}}(g)v$ for the action of g. There is an obvious standard basis given by the $\{[g]\}_{g\in G}$, which we shall now write $\{\boldsymbol{g}\}_{g\in G}$. We shall also use $\{\boldsymbol{g}^* := \frac{1}{|G|}\boldsymbol{g^{-1}}\}_{g\in G}$, which has the property (in $\mathbb{C}[G]$) that $\sum_{g\in G} \boldsymbol{g}\boldsymbol{g}^* = 1$.

Next, choose bases $\{e_j^\lambda\}_{j=1}^{n_\lambda}$ for each V_λ. By Artin-Wedderburn and writing endomorphisms of V_λ with respect to these bases, we have *ring* isomorphisms

$$\text{(III.D.3)} \qquad \mathbb{C}[G] \cong \times_{\lambda=1}^{r} \mathrm{End}_{\mathbb{C}}(V_\lambda) \cong \times_{\lambda=1}^{r} M_{n_\lambda}(\mathbb{C}).$$

To make this more explicit, write $(M_{ij}^\lambda(g)) \in M_{n_\lambda}(\mathbb{C})$ for the matrices of the action of g:

$$\text{(III.D.4)} \qquad g.e_j^\lambda =: \sum_{i=1}^{n_\lambda} M_{ij}^\lambda(g) e_i^\lambda.$$

Denoting $M(g) := (M^1(g), \ldots, M^r(g)) \in \times_{\lambda=1}^{r} M_{n_\lambda}(\mathbb{C})$, we can extend the map $g \mapsto M(g)$ $\mathbb{C}$-linearly to $\mathbb{C}[G]$, which recovers (III.D.3).

The upshot is that we may think of $\mathbb{C}[G]$ as comprising all r-tuples of matrices

$$M = (M^1, \ldots, M^r) \in \times_{\lambda=1}^r M_{n_\lambda}(\mathbb{C}),$$

or if you prefer, giant block matrices $\mathrm{diag}(M^1, \ldots, M^r) \in M_{\sum n_\lambda}(\mathbb{C})$. The action of such block matrices (think $\mathbb{C}[G]$) on themselves (think V_{reg}) by left multiplication makes the decomposition of V_{reg} into n_λ copies (columns) of each n_λ-dimensional representation quite concrete. This equivalence also tells us that there is another basis of V_{reg}: recalling that $\mathbf{e}_{ij}$ is the matrix with $(i,j)^{\mathrm{th}}$ entry 1 and all other entries 0, let $\mathbf{e}_{ij}^\lambda \in \times_{\lambda=1}^r M_{n_\lambda}(\mathbb{C})$ denote the r-tuple whose λ^{th} entry is $\mathbf{e}_{ij}$ and whose other entries are $\mathbf{0}$. Then $\{\mathbf{e}_{ij}^\lambda\}_{\lambda,i,j}$ is a basis of V_{reg}. What is the relation to $\{\boldsymbol{g}\}_{g\in G}$?

Fourier inversion formula.

To find this relation, first notice that any $\varphi \in \mathrm{Hom}_{\mathbb{C}}(V_\mu, V_\lambda)$ can be "averaged over G" to yield

$$\tilde{\varphi} := \tfrac{1}{|G|} \textstyle\sum_{g\in G} g^{-1} \circ \varphi \circ g \in \mathrm{Hom}_{\mathbb{C}[G]}(V_\mu, V_\lambda).$$

(Indeed, $\tilde{\varphi}$ intertwines the action of G since for $\gamma \in G$, reindexing by $g' = g\gamma$ yields $\tilde{\varphi}(\gamma.v) = \sum_g g^{-1}.\varphi(g\gamma.v) = \gamma \sum_{g'} (g')^{-1}.\varphi(g'.v) = \gamma.\tilde{\varphi}(v)$.) Writing this with respect to the bases gives ${}_{e^\lambda}[\varphi]_{e^\mu} =: (\varphi_{ij})$ and[3]

$$\tilde{\varphi}_{i\ell} = \textstyle\sum_{g\in G} \sum_{j,k} M_{ij}^\lambda(g^*)\varphi_{jk} M_{k\ell}^\mu(g). \tag{III.D.5}$$

By Schur's lemma, $\mathrm{Hom}_{\mathbb{C}[G]}(V_\mu, V_\lambda)$ is $\{0\}$ if $\lambda \neq \mu$ and $\mathbb{C}$ if $\lambda = \mu$ (again, the only division algebra over $\mathbb{C}$ is $\mathbb{C}$ itself). Accordingly, we have $\tilde{\varphi}_{i\ell} = C_{\mu,\varphi}\delta_{i\ell}\delta_{\lambda\mu}$ for some constant $C_{\mu,\varphi}$ (depending on μ and φ and only defined if $\lambda = \mu$).

When $\lambda = \mu$, we can calculate $C_{\mu,\varphi}$ by remembering that the trace $\sum_i B_{ii}$ of a matrix (B_{ij}) is invariant under conjugation:

$$\mathrm{tr}(A^{-1}BA) = \mathrm{tr}(B). \tag{III.D.6}$$

[3]Here I am writing $g^* = \frac{1}{|G|}[g^{-1}] \in \mathbb{C}[G]$, which we can insert in $M(\cdot)$ by the $\mathbb{C}$-linear extension mentioned above.

Since $M^{\mu}(g^{*}) = \frac{1}{|G|}(M^{\mu}(g))^{-1}$, we have

$$C_{\mu,\varphi}\mathrm{id}_{V_{\mu}} = [\tilde{\varphi}]_{e^{\mu}} = \tfrac{1}{|G|}\textstyle\sum_{g}(M^{\mu}(g))^{-1}[\varphi]_{e^{\mu}}M^{\mu}(g).$$

Applying trace now yields $n_{\mu}C_{\mu,\varphi} = \mathrm{tr}(\tilde{\varphi}) = \mathrm{tr}(\varphi)$, so that $C_{\mu,\varphi} = \frac{1}{n_{\mu}}\mathrm{tr}(\varphi)$.

In particular, taking $[\varphi]_{e^{\mu}} = \mathbf{e}^{\mu}_{\alpha\beta}$, i.e. $\varphi_{jk} = \delta_{j\alpha}\delta_{k\beta}$, yields $\mathrm{tr}(\varphi) = \delta_{\alpha\beta}$ hence $C_{\mu,\varphi} = \frac{1}{n_{\mu}}\delta_{\alpha\beta}$. Putting everything together, we get

$$\begin{aligned}\tfrac{1}{n_{\mu}}\delta_{\alpha\beta}\delta_{i\ell}\delta_{\lambda\mu} &= \textstyle\sum_{g}\sum_{j,k} M^{\lambda}_{ij}(g^{*})\delta_{j\alpha}\delta_{k\beta}M^{\mu}_{k\ell}(g)\\ &= \textstyle\sum_{g} M^{\lambda}_{i\alpha}(g^{*})M^{\mu}_{\beta\ell}(g)\\ &= M^{\mu}_{\beta\ell}\left(\textstyle\sum_{g} M^{\lambda}_{i\alpha}(g^{*})[g]\right)\end{aligned} \tag{III.D.7}$$

as $M^{\mu}_{\beta\ell}$ is $\mathbb{C}$-linear. Since $\mathbf{e}^{\lambda}_{\alpha i} \in \mathbb{C}[G]$ is by definition the element with $M^{\mu}_{\beta\ell}(\mathbf{e}^{\lambda}_{\alpha i}) = \delta_{\alpha\beta}\delta_{i\ell}\delta_{\lambda\mu}$, we arrive at the *Fourier inversion formula*

$$\sum_{g\in G} M^{\lambda}_{i\alpha}(g^{*})\boldsymbol{g} = \frac{1}{n_{\lambda}}\mathbf{e}^{\lambda}_{\alpha i} \tag{III.D.8}$$

in V_{reg}. This gives the desired relation between the two bases.

Character theory.

Recall that a *character of a group* is a homomorphism from G to $\mathbb{F}^{*}$ for some field $\mathbb{F}$. Taking $\mathbb{F} = \mathbb{C}$, since $\mathbb{C}^{*} \cong \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$ we can think of these as the 1-dimensional representations of G, or equivalently (since trace is the identity on 1×1 matrices), as their traces. This motivates the following more general notion for higher-dimensional representations:

III.D.9. DEFINITION. Let (V, π) be a representation of a finite group G over $\mathbb{C}$. (That is, $\pi\colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is the homomorphism through which G acts.) The **character of V** is the $\mathbb{C}$-valued function $\chi_{V}(g) := \mathrm{tr}(\pi(g))$ on G.[4] In particular, for the irreps V_{λ} we have

$$\chi_{\lambda}(g) := \chi_{V_{\lambda}}(g) = \mathrm{tr}(M^{\lambda}(g)) = \textstyle\sum_{i=1}^{n_{\lambda}} M^{\lambda}_{ii}(g).$$

[4]That is, we take the trace of the matrix of $\pi(g)$ with respect to any basis; this is independent of the choice of basis by (III.D.6).

I should remark right away that these are no longer homomorphisms from G to $\mathbb{C}^*$. (They don't intertwine multiplication, and can take the value 0.) But they have many spectacular properties which make them an indispensable tool for studying representations of G.

III.D.10. PROPOSITION. *Let V be a representation of G. Then*
(i) $\chi_V(1) = \dim V$.
(ii) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.
(iii) χ_V *is a **class function**, i.e. it is constant on conjugacy classes of G.*
(iv) *If W is another representation, then* $\chi_{V \oplus W} = \chi_V + \chi_W$.
(v) *If W is another representation, then* $\chi_{V \otimes W} = \chi_V \chi_W$.

PROOF. (i) follows from $\pi(1) = \mathrm{id}_V$, and (iii) from (III.D.6) since $\pi(\gamma g \gamma^{-1}) = \pi(\gamma)\pi(g)\pi(\gamma)^{-1}$. For (iv), just note that for block-diagonal matrices $\mathrm{tr}(\mathrm{diag}(M_1, M_2)) = \mathrm{tr}(M_1) + \mathrm{tr}(M_2)$; (v) is HW.

Finally, each $g \in G$ has finite order, so (the matrix of) $\pi(g)$ is diagonalizable[5] with root-of-1 eigenvalues ζ_i. Since each $\zeta_i^{-1} = \overline{\zeta}_i$, we have $\pi(g^{-1}) = \overline{\pi(g)}$ and taking traces gives (ii). □

III.D.11. EXAMPLE. Any symmetric group $\mathfrak{S}_n$ has two obvious 1-dimensional irreps, given by $\mathbf{1}$ (trivial) and **sgn**. There is an obvious n-dimensional representation U, given by letting the permutation act tautologically on the standard basis $\{e_i\}_{i=1}^n$ of $\mathbb{C}^n$. This is not irreducible, because it contains a copy of the trivial representation spanned by $e_1 + \cdots + e_n$. So it also contains a direct-sum complement **st**, the $(n-1)$-dimensional **standard representation**, which is just the subspace comprising vectors $\sum_i a_i e_i$ with $\sum_i a_i = 0$.

The character of U is computed by observing that the trace of a permutation (matrix) is the number of fixed elements, i.e. the number of "1-cycles" in the cycle-structure, which we write as F_σ. Therefore $\chi_{\mathbf{st}} = \chi_U - \chi_{\mathbf{1}}$ is given by $\chi_{\mathbf{st}}(\sigma) = F_\sigma - 1$.

Notice that (III.D.7) looks a bit like an orthogonality relation for matrix entries in the regular representation. To make this actually

[5]In characteristic 0, no Jordan block (of dimension > 1) has finite order.

true, one needs to construct a G-invariant inner-product (i.e. positive-definite Hermitian form) on each V_λ and choose the $\{e_i^\lambda\}$ to be orthonormal bases. But we're going to do something a bit simpler: define an inner product on $\mathbb{C}$-valued functions on G by

$$\langle \phi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)}.$$

Denoting the conjugacy classes of G by $\{C_\ell\}_{\ell=1}^N$, we have the

III.D.12. THEOREM (First Orthogonality Relation). *The characters of distinct irreps are orthogonal:* $\langle \chi_\mu, \chi_\lambda \rangle = \delta_{\mu\lambda}$*; more explicitly, we have*

$$\textstyle\sum_{\ell=1}^N |C_\ell| \chi_\mu(C_\ell)\overline{\chi_\lambda(C_\ell)} = |G|\delta_{\mu\lambda}.$$

PROOF. We have

$$\begin{aligned}\textstyle\sum_{\ell=1}^N |C_\ell|\chi_\mu(C_\ell)\overline{\chi_\lambda(C_\ell)} &= \textstyle\sum_{g\in G} \chi_\mu(g)\overline{\chi_\lambda(g)} \\ &= \textstyle\sum_{g\in G} \chi_\lambda(g^{-1})\chi_\mu(g) \\ &= \textstyle\sum_{g\in G} (\sum_{i=1}^{n_\lambda} M_{ii}^\lambda(g^{-1}))(\sum_{j=1}^{n_\mu} M_{jj}^\mu(g)) \\ &= |G| \textstyle\sum_{i,j} \left(\sum_{g\in G} M_{ii}^\lambda(g^*) M_{jj}^\mu(g) \right)\end{aligned}$$

which by (III.D.7) equals $\frac{|G|}{n_\mu}\sum_{i,j}\delta_{ij}\delta_{ij}\delta_{\lambda\mu}$. If $\lambda = \mu$, this becomes $\frac{|G|}{n_\mu}\sum_{i,j=1}^{n_\mu}\delta_{ij} = |G|$; so we obtain finally $|G|\delta_{\lambda\mu}$. ☐

Since orthogonal sets are independent, this has the

III.D.13. COROLLARY. *Given V, W representations of G, we have $V \cong W$ (as representations)* $\iff \chi_V = \chi_W$.

PROOF. Write $V \cong \oplus_\lambda V_\lambda^{\oplus p_\lambda}$ and $W \cong \oplus_\lambda V_\lambda^{\oplus q_\lambda}$. Then $\chi_V = \sum_\lambda p_\lambda \chi_\lambda$ and $\chi_W = \sum_\lambda q_\lambda\chi_\lambda$ by III.D.10(iv). Since the $\{\chi_\lambda\}$ are independent in the vector space of class functions, we have $\chi_V = \chi_W \iff p_\lambda = q_\lambda$ $(\forall\lambda)$. ☐

III.D.14. COROLLARY. (i) *The multiplicity of V_μ in V is* $\langle\chi_\mu, \chi_V\rangle$.
(ii) *A representation V of G is irreducible* $\iff \langle \chi_V, \chi_V\rangle = 1$.

PROOF. If $V \cong \oplus_\lambda V_\lambda^{\oplus p_\lambda}$, then by III.D.10(iii) and III.D.12 we have $\langle\chi_\lambda, \chi_V\rangle = \sum_\lambda p_\lambda\delta_{\lambda\mu} = p_\lambda$ and $\langle\chi_V,\chi_V\rangle = \sum_\lambda p_\lambda^2$. ☐

III.D.15. EXAMPLE. For $G = \mathfrak{S}_n$, I claim that the "standard representation" **st** defined in III.D.11 is irreducible. Consider the action of G on $\mathsf{X} := \{1, \ldots, n\}$ and $\mathsf{X} \times \mathsf{X}$; then the number of fixed points for $\sigma \in G$ is $\mathsf{X}^\sigma = F_\sigma$ resp. $(\mathsf{X} \times \mathsf{X})^\sigma = F_\sigma^2$.

On the other hand, the number of orbits is $|\mathsf{X}/G| = 1$ resp. $|(\mathsf{X} \times \mathsf{X})/G| = 2$, because the action of G on $\{1, \ldots, n\}$ is doubly transitive. Burnside's Lemma [**Algebra I**, II.N.2] immediately tells us that $\sum_{\sigma \in G} F_\sigma = |G|$ and $\sum_{\sigma \in G} F_\sigma^2 = 2|G|$. This yields

$$\langle \chi_{\mathbf{st}}, \chi_{\mathbf{st}} \rangle = \tfrac{1}{|G|} \textstyle\sum_{\sigma \in G} (F_\sigma - 1)^2 = \tfrac{1}{|G|} \left(\sum_\sigma F_\sigma^2 - 2 \sum_\sigma F_\sigma + \sum_\sigma 1 \right) = 1,$$

and so the claim follows from III.D.14(ii).

Next, look at $Z \subset \mathbb{C}[G]$, the center of the group ring: we shall compute its dimension in two different ways.

(1) First, think in terms of $\times_{\lambda=1}^r M_{n_\lambda}(\mathbb{C})$. The minimal central idempotents $\{\sum_{i=1}^{n_\lambda} \mathbf{e}_{ii}^\lambda\}_{\lambda=1}^r$ span Z, since the center of each block $M_{n_\lambda}(\mathbb{C})$ is just $\mathbb{C}\mathrm{id}_{n_\lambda}$. So $\dim_{\mathbb{C}} Z = r$, the number of irreps.
(2) Next, think in terms of the group ring $\mathbb{C}[G]$. The elements $\{\sum_{g \in C_\ell} [g]\}_{\ell=1}^N$ are obviously fixed under conjugation by each $[\gamma]$, hence belong to Z. Moreover, anything in Z must be invariant under $\frac{1}{|G|} \sum_g [g](\,\cdot\,)[g^{-1}]$, and this forces it to be a sum of these elements. So they are also a basis, and $\dim_{\mathbb{C}} Z = N$, the number of conjugacy classes.

Together with the independence of the $\{\chi_\lambda\}_{\lambda=1}^r$, this proves the

III.D.16. THEOREM. *The number of conjugacy classes in G is equal to the number of irreducible representations of G, i.e. $N = r$. Consequently, the χ_λ are a basis for the class functions on G.*

We can use this to establish yet another

III.D.17. THEOREM (Second Orthogonality Relation).

$$|C_\ell| \textstyle\sum_{\lambda=1}^r \chi_\lambda(C_k) \overline{\chi_\lambda(C_\ell)} = |G| \delta_{\ell k}.$$

PROOF. Define "indicator" class functions on G by $f_\ell(C_k) := \delta_{k\ell}$. We may write these in terms of the basis $\{\chi_\lambda\}$, viz. $f_\ell = \sum_\lambda \alpha_\lambda \chi_\lambda$.

Then

$$\begin{aligned}\alpha_\tau &= \sum_\lambda \alpha_\lambda \delta_{\lambda\tau} = \sum_\lambda \alpha_\lambda \langle \chi_\lambda, \chi_\tau \rangle = \langle f_\ell, \chi_\tau \rangle \\ &= \tfrac{1}{|G|} \sum_{h\in C_\ell} \overline{\chi_\tau(h)} = \tfrac{|C_\ell|}{|G|} \overline{\chi_\tau(C_\ell)}\end{aligned}$$

$\implies f_\ell(g) = \sum_\lambda \alpha_\lambda \chi_\lambda(g) = \frac{|C_\ell|}{|G|} \sum_\lambda \overline{\chi_\lambda(C_\ell)} \chi_\lambda(g)$. Taking $g \in C_k$ gives the result. □

III.D.18. COROLLARY. (i) *The* $n_\lambda := \dim V_\lambda$ *satisfy* $\sum_{\lambda=1}^r n_\lambda^2 = |G|$. (ii) *If* G *is abelian, then every* $n_\lambda = 1$*, i.e. all irreps are 1-dimensional and given by "characters of* G*".*

PROOF. Of course, we already know (i) by Artin-Wedderburn. But it is nice to see it confirmed by character theory: taking $C_\ell = C_k = \{1\}$ in III.D.17, we get $\sum_\lambda (\chi_\lambda(1))^2 = |G|$ (now apply III.D.10(i)). If G is abelian, every conjugacy class has one element, so there are $|G|$ of them. By III.D.16, there are $|G|$ irreps and so $r = |G|$ in (i), which forces all $n_\lambda = 1$. □

III.D.19. EXAMPLE. Take the (cyclic) abelian group $G = \mathbb{Z}_n$. Its representations are necessarily 1-dimensional, so their characters are characters of G, i.e. homomorphisms $G \to \mathbb{C}^*$. So the **character table** is simply

	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\cdots$	$\overline{n-1}$
χ_0	1	1	1	$\cdots$	1
χ_1	1	ζ_n	ζ_n^2	$\cdots$	$\bar{\zeta}_n$
$\vdots$	$\vdots$				$\vdots$
χ_{n-1}	1	$\bar{\zeta}_n$	$\bar{\zeta}_n^2$	$\cdots$	ζ_n

Usually the characters of irreps are numbered 1 to r, but here starting with 0 made more sense. The top line of the character table lists the conjugacy classes (which here are simply the elements of G), and the interior of the table lists the values taken by each character.

III.D.20. EXAMPLE. For a nonabelian group, consider $G = \mathfrak{S}_4$. It has five conjugacy classes, corresponding to the possible cycle structures. So there must be five irreps with characters $\chi_1, \ldots, \chi_5$. As with any symmetric group, there are two obvious 1-dimensional irreps (characters of G), given by $V_1 := \mathbf{1}$ (trivial) and $V_2 := \mathbf{sgn}$ (alternating). By III.D.15, we also have the standard irrep $V_3 := \mathbf{st}$, of dimension 3. The tensor product of any character of G with an irrep is always again irreducible, by III.D.10(v) and III.D.14(ii) (since multiplying each $\chi(g)$ by a root of unity doesn't change $\frac{1}{|G|}\sum_g |\chi(g)|^2 = 1$), and so $V_4 := \mathbf{st} \otimes \mathbf{sgn}$ is another 3-dimensional irrep.

This leaves V_5. The sum of squares of dimensions must satisfy $\sum_{\lambda=1}^5 n_\lambda^2 = |G| = 24$, from which $n_5 = 2$. We can now easily compute its character χ_5 by noticing that the regular representation has $\chi_{\mathbb{C}[G]}(1) = |G|$ and $\chi_{\mathbb{C}[G]}(g) = 0$ for $g \neq 1$. (This is a general fact. Why?) Since $\mathbb{C}[G] = V_1 \oplus V_2 \oplus V_3^{\oplus 3} \oplus V_4^{\oplus 3} \oplus V_5^{\oplus 2}$, we obtain the final row of the character table for $\mathfrak{S}_4$

	$\mathbf{1}$	$(\cdot\cdot)$	$(\cdot\cdot\cdot)$	$(\cdot\cdot\cdot\cdot)$	$(\cdot\cdot)(\cdot\cdot)$
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	1	-1
χ_5	2	0	-1	0	2

from $\chi_5 = \frac{1}{2}\chi_{\mathbb{C}[G]} - \frac{1}{2}\chi_1 - \frac{1}{2}\chi_2 - \frac{3}{2}\chi_3 - \frac{3}{2}\chi_4$. One can show that V_5 is obtained by composing the quotient map $\mathfrak{S}_4 \twoheadrightarrow \mathfrak{S}_4/V_4 \cong \mathfrak{S}_3$ with the standard representation of $\mathfrak{S}_3$. (Can you do it with characters?)

III.D.21. REMARK. If we replace $\overline{\chi(g)}$ by $\chi(g^{-1})$, everything we have done in this section (over $\mathbb{C}$) works over a more general field $\mathbb{F}$, provided (a) char($\mathbb{F}$) does not divide $|G|$ and (b) $\mathbb{F}$ is a *splitting field for G*. The latter means that $\mathbb{F}[G]$ splits into absolutely irreducible representations – irreps which remain irreducible over $\bar{\mathbb{F}}$. (Failure of (b) is equivalent to one or more of the matrix rings in Artin-Wedderburn no longer having coefficients in $\mathbb{F}$, but in a larger field or division algebra.) The HW problems feature instances both where $\mathbb{Q}$ is, and where $\mathbb{Q}$ is not, a splitting field for G in this sense.

Induced representations.

I'd like to briefly mention a very useful construction of representations of groups from those of its subgroups, which can be described very nicely in terms of group algebras. So let $H \leq G$ be any subgroup of our finite group G.

We first point out the obvious fact that a representation $\pi\colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$ can be composed with the inclusion homomorphism $H \hookrightarrow G$, to yield the **restriction**

$$\mathrm{Res}^G_H \pi\colon H \to \mathrm{Aut}_{\mathbb{C}}(V)$$

of π to H, which we can informally write as $\mathrm{Res}(V)$ when G and H are understood.

Next, suppose $W \subset V$ is an H-invariant subspace, i.e. a subrepresentation $W \subset \mathrm{Res}(V)$ (of H). Notice that $g.W := \pi(g)(W)$ depends only on the coset $gH =: \gamma$, so it makes sense to write $\gamma.W$. We say that V is **induced** by W if $V = \oplus_{\gamma \in G/H} \gamma.W$.

Now begin from the opposite end of things: suppose we are given a representation $\eta\colon H \to \mathrm{Aut}_{\mathbb{C}}(W)$ of H. Then there exists a unique representation of G induced by W, called $\mathrm{Ind}^G_H \eta$, or informally $\mathrm{Ind}(W)$. In fact, it is given by

$$\mathrm{Ind}(W) := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W, \tag{III.D.22}$$

where the tensor product *over a ring* means that $[g] \otimes hw = [gh] \otimes w$, and the action of G is by $g.([g'] \otimes w) = [gg'] \otimes w$. Note that the dimension of (III.D.22) is $|G/H| \dim(W)$.

To make this completely explicit, let g_γ be coset representatives. For each coset $\gamma \in G/H$, let W_γ be a copy of W whose elements are formally written $g_\gamma w$ (with $w \in W$), and set $V := \oplus_{\gamma \in G/H} W_\gamma$. Then every element of V may be written uniquely as $v = \sum_\gamma g_\gamma w_\gamma$, and we define the action of G on V by $g.(g_\gamma w_\gamma) := g_{\gamma'}(hw_\gamma)$ if $gg_\gamma = g_{\gamma'}h$.

The first main theorem on induced representations is **Frobenius reciprocity**. Given a representation W of H and U of G, it reads

$$\langle \chi_{\mathrm{Ind}(W)}, \chi_U \rangle_G = \langle \chi_W, \chi_{\mathrm{Res}(U)} \rangle_H \tag{III.D.23}$$

in terms of the inner products on functions on G and H respectively. If W and U are both irreducible, then (together with III.D.14(i)) this has the immediate corollary that *the multiplicity of U in* $\mathrm{Ind}(W)$ *equals the multiplicity of W in* $\mathrm{Res}(U)$.

III.D.24. EXAMPLE. The alternating group $\mathfrak{A}_5$ possesses a unique 5-dimensional irrep. One way to construct it is by taking $H \leq \mathfrak{A}_5$ to be a copy of $\mathfrak{A}_4$, and applying $\mathrm{Ind}_H^{\mathfrak{A}_5}$ one of the two nontrivial characters (1-diml irreps) of $\mathfrak{A}_4$. Another way is to construct it is related to the action of $\mathfrak{S}_5$ on its 6 Sylow 5-subgroups, and is considered in the HW.

Group cohomology.

Finally, to wrap up representation theory, I describe one interesting thing you can do with representations of G. In fact, a little more generally, let M be an abelian group on which G acts by automorphisms: so it could be a representation, or it could be the multiplicative group of a field.

Define $C^k := C^k(G, M)$ ($k \geq 0$) to be the group of all functions $\varphi \colon G^{\times k} \to M$ (where $G^{\times 0} := \{1\}$, $G^{\times 1} := G$, $G^{\times 2} := G \times G$, etc.). We also need a *differential* $d \colon C^k \to C^{k+1}$, which is given by the formula

$$\begin{aligned}(d\varphi)(g_1, \ldots, g_{k+1}) :&= g_1.\varphi(g_2, \ldots, g_{k+1}) \\ &+ \textstyle\sum_{i=1}^{k} (-1)^i \varphi(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{k+1}) \\ &+ (-1)^{k+1} \varphi(g_1, \ldots, g_k).\end{aligned}$$

You can check that $d \circ d = 0$, so that $\mathrm{im}(d) \subseteq \ker(d)$ in each C^k. We say the sequence (or "cochain complex")

$$0 \to C^0 \xrightarrow{d} C^1 \to \cdots \to C^{k-1} \xrightarrow{d} C^k \xrightarrow{d} C^{k+1} \to \cdots$$

is *exact* at the k^{th} term if this inclusion is an equality. Cohomology measures the failure of our complex $(C^\bullet, d)$ to be exact.

III.D.25. DEFINITION. The k^{th} **cohomology group** of G with coefficients in M is

$$H^k(G,M) := \frac{\ker\{d\colon C^k \to C^{k+1}\}}{\mathrm{im}\{d\colon C^{k-1} \to C^k\}}.$$

Let's consider two special cases. For $k=0$, we note that $C^0 = M$ and the differential sends $m \in M$ to $dm \in C^1$, which is a function on G defined by $(dm)(g) := g.m - m$. So the kernel consists of those $m \in M$ with $g.m = m$, i.e.

$$H^0(G,M) = M^G \qquad \text{(III.D.26)}$$

are the G-invariants. For $k=1$, the elements of C^1 are functions $\varphi\colon G \to M$, and the differential reads $(d\varphi)(g_1,g_2) = g_1.\varphi(g_2) - \varphi(g_1g_2) + \varphi(g_1)$. So the cohomology is

(III.D.27)
$$H^1(G,M) = \frac{\{\varphi\colon G \to M \mid \varphi(gg') = g.\varphi(g') + \varphi(g)\}}{\{\varphi\colon G \to M \mid \varphi(g) = g.m - m \text{ for some } m\}},$$

the so-called *crossed homomorphisms* modulo *principal crossed homomorphisms*. If the action of G on M is trivial, then (III.D.26) is M and (III.D.27) just becomes $\mathrm{Hom}(G,M)$.

More intriguing is the case where G is the Galois group of an extension L/K, acting on $M = L^*$ through the automorphisms. (Here the group operation on M will be written multiplicatively.) In that case, (III.D.26) is evidently K^*, but what about H^1? Well, the Lemma associated with Hilbert's Theorem 90 says, verbatim:[6] *let* $\varphi\colon G \to L^*$ *be a map satisfying* $\varphi(gg') = g(\varphi(g'))\varphi(g)$. *Then there exists* $\ell \in L^*$ *such that* $\varphi(g) = g(\ell)/\ell$. So the crossed homomorphisms are all principal, and

$$H^1(G,L^*) = \{0\}, \qquad \text{(III.D.28)}$$

thus revealing Hilbert's theorem as *the* foundational result in Galois cohomology.

[6] I have made slight changes in the notation, including replacing ℓ_0 by $\ell := \ell_0^{-1}$.