IV.B. Chain conditions

In this short section we won't assume R is commutative, since it connects both to the material on representation theory just covered and to the material ahead. (Some of it will look familiar; in fact, IV.B.12 is just #4 from Problem Set 8.) Let M be a left or right R-module.

IV.B.1. DEFINITION. (i) M is **Noetherian** if the ACC holds: for every ascending chain $M_1 \subset M_2 \subset \cdots (\subset M)$ of submodules, there exists $n \in \mathbb{N}$ such that $M_i = M_n$ for every $i \geq n$.

(ii) M is **Artinian** if the DCC holds: for every descending chain $M \supset M_1 \supset M_2 \supset \cdots$ of submodules, there exists $n \in \mathbb{N}$ such that $M_i = M_n$ for every $i \geq n$.

In either case, we say that the chain *stabilizes* at n.

IV.B.2. EXAMPLE. (i) As a $\mathbb{Z}$-module, $\mathbb{Z}$ is Noetherian (because submodules have finite index) but not Artinian ($M_k := (p^k)$ violates the DCC).

(ii) As a $\mathbb{Z}$-module, $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ is Artinian (because proper submodules have finite order) but not Noetherian ($M_k := (\frac{1}{p^k})$ violates the ACC).

IV.B.3. DEFINITION. R is said to be left [resp. right] Artinian or Noetherian (as a ring) if ${}_R R$ [resp. R_R] is Artinian or Noetherian. We remove the left/right moniker if it is both left and right Artinian or Noetherian.

IV.B.4. EXAMPLES. (i) Any division ring D is trivially Noetherian and Artinian: there are no proper nontrivial left or right ideals.

(ii) Any matrix ring $M_n(D)$ over a division ring is Noetherian and Artinian, cf. IV.B.13.

(iii) Any commutative PID is Noetherian. [Proof: given a sequence $I_1 \subset I_2 \subset \cdots$, i.e. $(r_1) \subset (r_2) \subset \cdots$, of ideals, take the union I. This is an ideal, so $I = (r)$. But $r \in (r_n)$ for some n, so $I \subset (r_n)$ hence $I_i = I_n$ for $i \geq n$.]

(iv) Every ring of integers in a number field is Noetherian. [Proof: recall that associated to each ideal $I \subset \mathcal{O}_K$ is a norm $\mathfrak{N}(I) := |\mathcal{O}_K/I| \in \mathbb{N}$; it has the property that $I_1 \subset I_2 \implies N(I_1) > N(I_2)$, with equality iff $I_1 = I_2$. Since a decreasing sequence of natural numbers must eventually stabilize, so must any ascending chain of ideals.]

IV.B.5. REMARK. The ACC and DCC may be rephrased in terms of *every nonempty set* $\mathfrak{M}$ *of submodules of* M *having a maximal and minimal element*. If the latter property holds, then any ascending [resp. descending] chain stabilizes: simply take $\mathfrak{M}$ to *be* the chain $\{M_i\}_{i\in\mathbb{N}}$. The maximal [resp. minimal] element must be a *member* of the chain, i.e. some M_n, and then the chain stabilizes at n.

Conversely, assuming e.g. the DCC, suppose that some nonempty set of modules $\mathfrak{M}$ has no minimal element. For each $M' \in \mathfrak{M}$, we may then choose $M'' \in \mathfrak{M}$ such that $M' \supsetneq M''$, defining a map $f\colon \mathfrak{M} \to \mathfrak{M}$. (This uses the Axiom of Choice.) Now pick $M_0 \in \mathfrak{M}$, and define M_i recursively by $M_i := f(M_{i-1})$; this gives $M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$, in contradiction to the DCC.

IV.B.6. LEMMA. *Given an exact sequence* $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ *of* R*-modules,* B *is Noetherian* [*resp. Artinian*] $\iff$ A *and* C *are Noetherian* [*resp. Artinian*].

PROOF (FOR NOETHERIAN). $(\implies)$: An ascending chain of submodules of A or C gives an ascending chain in B via f resp. g^{-1}, which then must stabilize (since B is Noetherian). This forces the chains in A or C to stabilize as well, and so A and C satisfy the ACC.

$(\impliedby)$: Let $B_1 \subset B_2 \subset \cdots$ be an ascending chain in B. We need to show that it stabilizes. Set $A_i := f^{-1}(f(A) \cap B_i)$ and $C_i := g(B_i)$; then writing $f_i := f|_{A_i}$ and $g_i := g|_{B_i}$,

$$0 \to A_i \overset{f_i}{\to} B_i \overset{g_i}{\to} C_i \to 0$$

is exact since $\ker(g_i) = \ker(g) \cap B_i = f(A) \cap B_i = f(A_i)$. Now the $\{A_i\}$ and $\{C_i\}$ stabilize at some $n \in \mathbb{N}$, yielding for each $i \geq n$ a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \imath} & & \| & & \\ 0 & \longrightarrow & A_i & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & 0. \end{array}$$

Given $\beta \in B_i$, let $\gamma = g_i(\beta) \in C_i = C_n$, and pick $\beta' \in g_n^{-1}(\gamma) \subset B_n$. Since g_i sends $\beta - \imath(\beta') \mapsto 0$, there exists $\alpha \in A_i = A_n$ such that $\beta - \imath(\beta') = f_i(\alpha) = \imath(f_n(\alpha))$. But then $\beta = \imath(\beta' + f_n(\alpha))$, which shows $\imath$ is surjective and $B_i = B_n$. □

IV.B.7. REMARK. (i) The lemma has the following immediate consequence: given a submodule $N \subset M$, if M satisfies ACC or DCC, then so do N and M/N (use the exact sequence $0 \to N \to M \to M/N \to 0$).

(ii) Similarly, $M_1 \oplus \cdots \oplus M_n$ satisfies ACC/DCC iff the $\{M_i\}$ do (use $0 \to M' \to M' \oplus M'' \to M'' \to 0$ and induction).

(iii) The argument in the lemma's proof can be generalized (see the HW) to prove the *five-lemma*: suppose we are given a commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow{\scriptstyle \jmath} & & \| & & \downarrow & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 & \longrightarrow & 0 \end{array}$$

of R-modules, with exact rows; then $\jmath$ is an isomorphism.

IV.B.8. THEOREM. *If R is left* [*resp. right*] *Noetherian or Artinian, then so is every finitely generated left* [*resp. right*] *R-module.*

PROOF. To say that a left [resp. right] R-module M is finitely generated is to say that there is a surjective left [resp. right] R-module homomorphism $g\colon R^{\oplus n} \twoheadrightarrow M$. That is, $M \cong R^{\oplus n}/\ker(g)$. The hypothesis is that, as a left [resp. right] R-module, R itself satisfies a chain condition (ACC or DCC). By IV.B.7(ii), $R^{\oplus n}$ satisfies the chain condition; and then, by IV.B.7(i), so does its quotient M. □

The next result gives an important characterization of Noetherianity:

IV.B.9. THEOREM. *An R-module satisfies the ACC $\iff$ every submodule is finitely generated.*

PROOF. $\underline{(\implies)}$: Given an R-submodule $N \subset M$, where M satisfies ACC. By IV.B.5, the set $\mathfrak{M}$ of f.g. submodules of N contains a maximal element $N_0 = R\langle \eta_1, \ldots, \eta_n \rangle$. Take any $\eta \in N$. By maximality of N_0, $R\langle \eta_1, \ldots, \eta_n, \eta \rangle = N_0$; hence $N_0 = N$ and N is f.g.

$\underline{(\impliedby)}$: Given a chain $M_1 \subset M_2 \subset \cdots$, $M_\infty := \cup_{j>0} M_j$ is a submodule of M hence f.g., by (say) $\mu_1, \ldots, \mu_n$. Each of these is in some M_i, and so $M_\infty \subset M_{i_0}$ for some i_0. That is, the chain stabilizes at i_0. $\square$

If we consider R as a (say, left) module over itself, then this says R is left Noetherian iff its left ideals are all finitely generated. In particular:

IV.B.10. COROLLARY. *Suppose R is commutative. Then R is Noetherian $\iff$ every ideal is finitely generated.*

IV.B.11. REMARK. With some effort, we checked *both* the ACC and finite generation of ideals for number rings. Depending on your outlook, it's either reassuring or unpleasant to realize, in retrospect, that we only needed to check one or the other.

IV.B.12. THEOREM. *An R-module M has a composition series $\iff$ M satisfies ACC and DCC.*

PROOF. $\underline{(\implies)}$: By the 1st Isomorphism Theorem, we have the

Fact 1: *If $N \supseteq N'$ and M' are submodules of M, with $N \cap M' = N' \cap M'$ $(=: K)$ and $N/K = N'/K$, then $N = N'$.*

So if M has a composition series $M = M_0 \supset M_1 \supset \cdots \supset M_n = \{0\}$, and $N \supseteq N'$, then by induction Fact 1 implies

Fact 2: *If $N \supseteq N'$ and $\frac{N \cap M_i}{N \cap M_{i+1}} = \frac{N' \cap M_i}{N' \cap M_{i+1}}$ for all i, then $N = N'$.*

But now observe that $\mathrm{Gr}_i N := \frac{N \cap M_i}{N \cap M_{i+1}} = \frac{(N \cap M_i) + M_{i+1}}{M_{i+1}}$ is a submodule of $\frac{M_i}{M_{i+1}}$, which is simple (by defintion of CS). So $\mathrm{Gr}_i N$ is either

$\{0\}$ or $\frac{M_i}{M_{i+1}}$, and we set $\ell(N) := \#\{i \mid \mathrm{Gr}_i N \neq \{0\}\}$. Considering any ascending or decending chain beginning at N, it is now clear that — since a *proper* inclusion requires (by Fact 2) an increase or decrease in $\ell(\cdot)$, which is bounded above (by n) and below (by 0) — the chain must stabilize. So ACC and DCC hold, i.e. M is Noetherian and Artinian.

$\underline{(\Longleftarrow)}$: Suppose M is Noetherian and Artinian, and let $N \subseteq M$ be a nonzero submodule. Then N has a maximal proper submodule by Noetherianity and IV.B.5. Now let $M_0 := M$, M_1 be a maximal proper submodule, M_2 be a maximal proper submodule of that, and so on (which we can do by the Axiom of Choice, as in IV.B.5). By construction, this decending series can only stabilize at $\{0\}$, and stabilize it must, by the Artinian hypothesis. Moreover, the successive quotients are simple by the 1st Isomorphism Theorem; and so we get our composition series. □

IV.B.13. REMARK. In particular, if ${}_R R$ [resp. R_R] has a CS, then R is left [resp. right] Noetherian and Artinian. For instance, in view of the composition series constructed in the proof of III.B.2 (and its "transpose"), we conclude from IV.B.12 that $M_n(D)$ (D a division ring) is left and right Noetherian and Artinian.