Problem Set 10

Do (3)-(8), and your choice of (1) or (2).

(1) Let V be a $k[G]$-module and H a subgroup in G of finite index not divisible by $\mathrm{char}(k)$. Modify the proof of Maschke's theorem to show the following: if V is semisimple as a $k[H]$-module, then V is semisimple as a $k[G]$-module.

(2) (a) For $G = \mathfrak{S}_3$ (symmetric group), show that $\mathbb{Q}[G] \cong \mathbb{Q} \times \mathbb{Q} \times M_2(\mathbb{Q})$ and compute the central idempotents of $\mathbb{Q}[G]$ which give the decomposition of $\mathbb{Q}[G]$ into its simple components. (That is, use (III.D.8) to compute the elements $\sum_{i=1}^{n_\lambda} \mathbf{e}_{ii}^{\lambda}$.) (b) Do the same thing for the Klein 4-group.

(3) Let Q be the quaternion group. Find the irreducible representations of Q and product decomposition of $\mathbb{F}[Q]$, for $\mathbb{F} = \mathbb{R}$ (or $\mathbb{Q}$) and $\mathbb{C}$.

(4) Recall that $\mathfrak{S}_5$ acts transitively on its six Sylow 5-subgroups. This gives a 6-dim'l representation. Compute its character just on the "even conjugacy classes", which amounts to its character as a representation of $\mathfrak{A}_5$. Now it has an obvious fixed vector (i.e. copy of the trivial representation). Show that the complementary 5-dimensional subrepresentation is irreducible as a representation of $\mathfrak{A}_5$ (hence $\mathfrak{S}_5$) by using its character.

(5) Show that, over $\mathbb{Q}$, the alternating group $\mathfrak{A}_5$ has four irreducible representations ("irreps"), of dimensions $1, 4, 5, 6$ respectively. [Hint: look at how $\mathfrak{A}_5$ appears in the section on Burnside from last term. What is the trace of a rotation by 72 degrees?]

(6) Show that the 3-dimensional irrep $\mathbf{st} \otimes \mathbf{sgn}$ of $\mathfrak{S}_4$ is equivalent (i.e. isomorphic as $\mathbb{C}[\mathfrak{S}_4]$ modules) to the representation of $\mathfrak{S}_4$ as the group of rotational symmetries of the cube (or octahedron). [Hint: compute the character of the latter.]

(7) Show that the character of a tensor product of representations is the product of the characters. Use this together with III.D.14(i) to compute the decompositions of $\mathbf{st} \otimes \mathbf{W}$ and $\mathbf{st} \otimes \mathbf{st}$ into irreps of $\mathfrak{S}_4$.

(8) Suppose the character table of a finite group has the following two rows ($\zeta_3 = e^{\frac{2\pi \mathbf{i}}{3}}$)

1	1	1	ζ_3^2	ζ_3	ζ_3^2	ζ_3
2	-2	0	-1	-1	1	1

corresponding to characters of two irreps. (The first column gives the value on $\{1\}$, and the remaining columns the values on the other six conjugacy classes.) Determine the rest of the character table, including the orders of the group and its conjugacy classes.