PROBLEM SET 11

R is a commutative ring (with 1, as always in this course) throughout.

(1) Prove that an Artinian commutative domain R is a field. [Hint: to find an inverse for $a \neq 0$, consider $(a) \supset (a^2) \supset (a^3) \supset \cdots$.]

(2) If $\mathcal{S}$ is a multiplicative subset of a commutative ring R, show that (a) $S^{-1}(\text{Rad}I) = \text{Rad}(\mathcal{S}^{-1}I)$ and (b) $\mathcal{S}^{-1}R$ is Noetherian if R is Noetherian.

(3) Show that a commutative ring is local if and only if for all $r, s \in R$, $r + s = 1$ implies r or s is a unit.

(4) Let p be a prime in $\mathbb{Z}$; then (p) is a prime ideal. What can be said about the relationship between $\mathbb{Z}_p$ and the localization $\mathbb{Z}_{(p)}$? Describe $\mathbb{Z}_{(p)}$ as a subset of the rational numbers.

(5) Find the nilradical of $\mathbb{Z}_n$ ($n \in \mathbb{N}$).

(6) Prove the five-lemma (Remark IV.B.7(iii)).

(7) If every maximal ideal in R is of the form (c), for some $c \in R$ with $c^2 = c$, then R is Noetherian. [Hint: show that every primary ideal is maximal; use Cohen's theorem.]

(8) Show that in $\mathbb{Z}[x, y]$ the ideals (x^i, y^j) are all primary ideals with radical (x, y).

(9) Find a reduced primary decomposition for the ideal $I = (x^2, xy, 2)$ in $\mathbb{Z}[x, y]$ and determine the (prime) radicals of the primary ideals appearing in this decomposition.