PROBLEM SET 12

R is a commutative ring (with 1) throughout.

(1) If N is a P-primary submodule of an R-module M and $rx \in N$ ($r \in R$, $x \in M$), show that either $r \in P$ or $x \in N$.

(2) If M is an R-module and $x \in M$, the annihilator of x, denoted $\operatorname{ann}(x)$, is $\{r \in R \mid rx = 0\}$. (a) Show that $\operatorname{ann}(x)$ is an ideal. (b) Assuming $M \neq 0$, show that a maximal element of the set $\{\operatorname{ann}(x) \mid x \in M \backslash \{0\}\}$ of ideals is prime.

(3) Let R be Noetherian and $M \neq \{0\}$ an R-module. If P is prime, of the form $\operatorname{ann}(x)$ for some $x \in M$, then P is called an **associated prime** of M. (a) Using (2)(b), show that an associated prime exists. (b) If M satisfies the ACC, prove that there exist primes $P_1, \ldots, P_{r-1}$ and a sequence of submodules $M = M_1 \supset M_2 \supset \cdots \supset M_r = \{0\}$ such that $M_i / M_{i+1} = R/P_i$ for each $i < r$.

(4) Prove that $\mathbb{C}[z]$ is integrally closed (in its fraction field $\mathbb{C}(z)$), i.e. *normal*. [Hint: suppose you had a $\frac{P}{Q} \in \mathbb{C}(z)$, P and Q relatively prime in $\mathbb{C}[z]$, integral over $\mathbb{C}[z]$.]

(5) In lecture on Monday I'll describe *normalizing* the curve $y^2 = x^3 - x^2$ with a nodal singularity at $(0,0)$ by introducing the function $z = \frac{y}{x}$. If $R = \frac{\mathbb{C}[x,y]}{(y^2 - x^3 + x^2)}$, then let $S = \frac{\mathbb{C}[x,y,z]}{(y^2 - x^3 + x^2, z^2 - x + 1, z^3 + z - y)}$ with the natural map $\varphi \colon R \to S$. Show
(a) that $S \cong \mathbb{C}[z]$ (the coordinate ring of a complex line!).
(b) What geometric map does φ correspond to "pulling back functions" along? Use this to argue that φ is injective (or prove this by some other means).
(c) Show (e.g. using Chinese remainder) that $T = \frac{\mathbb{C}[x,y,z]}{(y^2 - x^3 + x^2, z^2 - x + 1)}$ is not a domain, so this can't be inside R's field of fractions.
(d) Use problem (4) to show that S is the integral closure of R in its fraction field.
[Hint: don't make this problem hard. It's all very simple calculations or trivial arguments.]

(6) Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. If the ideal $\mathfrak{m}/\mathfrak{m}^2$ in $R/\mathfrak{m}^2$ is generated by $\{a_1 + \mathfrak{m}^2, \ldots, a_n + \mathfrak{m}^2\}$, show that the ideal $\mathfrak{m}$ is generated in R by $\{a_1, \ldots, a_n\}$. [Hint: use Nakayama's Lemma (i) $\Longrightarrow$ (iv).]

Remark: In the notes, "associated primes $\{P_i\}$ of a reduced primary decomposition" of a submodule $A \subset B$ (of an R-module B) were defined in a different way (as the $\sqrt{Q_{A_i}}$ of the primary modules A_i whose intersection is A). With some additional work, one can show that these $\{P_i\}$ are precisely the associated primes of the R-module B/A (in the sense of exercise (3) here).