PROBLEM SET 3

(1) [Jacobson p. 229 #2] Construct a splitting field over $\mathbb{Q}$ of $x^5 - 2$. Find its degree over $\mathbb{Q}$.
(2) [Jacobson p. 229 #3] Determine a splitting field over $\mathbb{Z}_p$ of $x^{p^e} - 1$, $e \in \mathbb{N}$.
(3) Find splitting field extensions for $x^3 - 5$ over $\mathbb{Z}_7$, $\mathbb{Z}_{11}$ and $\mathbb{Z}_{13}$.
(4) Show that an algebraically closed field must be infinite.
(5) Suppose that $K(\alpha)/K$ is a simple extension and that α is transcendental over K. Show that $K(\alpha)$ is not algebraically closed.
(6) Let p be a prime number. By factorizing $x^{p-1} - 1$ over $\mathbb{Z}_p$, prove *Wilson's theorem*: i.e., show that $(p-1)! \underset{(p)}{\equiv} -1$. (I know it's a retread from Algebra I, but this gives a quicker proof and new perspective.)
(7) [Jacobson p. 234 #2] Let $f(x)$ be irreducible in $F[x]$, where F is of characteristic p. Show that $f(x)$ can be written as $g(x^{p^e})$, where $g(x)$ is irreducible and separable. Use this to show that every root of $f(x)$ has the same multiplicity p^e (in a splitting field). [Hint: use I.E.6 and I.E.8 from the Algebra II notes.]