PROBLEM SET 4

Hand in: 1 or 2; 3 or 4; and 5-10. (8 problems)

(1) [Jacobson p. 234 #4] Let F be imperfect of characteristic p. Show that $x^{p^e} - a$ is irreducible if $a \notin F^p$ and $e = 0, 1, 2, \ldots$.

(2) Let K be a field of positive characteristic.
(i) Show that K is perfect if and only if the Frobenius homomorphism is an automorphism.
(ii) If L/K is a totally inseparable extension (i.e. every element of $L \backslash K$ is inseparable), show that the minimal polynomial of any element of L over K is of the form $x^{p^n} - \alpha$, where $\alpha \in K$.

(3) Suppose that L/K is algebraic. Show that there is a greatest intermediate field $M(\subset L)$ such that M/K is normal.

(4) Suppose that L/K is finite, with normal closure L^c/L. Show that L/K is separable if and only if there are exactly $[L{:}K]$ embeddings of L into L^c fixing K.

(5) Suppose that L/K is a finite normal extension and that f is an irreducible polynomial in $K[x]$. Suppose that g and h are irreducible monic factors of f in $L[x]$. Show that there is an automorphism σ of L which fixes K such that $\sigma(g) = h$.

(6) Suppose that L/K is a Galois extension, $G = \mathrm{Aut}(L/K)$, and $\alpha \in L$. Show that $L = K(\alpha)$ if and only if the images of α under the elements of G are all distinct.

(7) Suppose that L/K is a Galois extension with Galois group $G = \{\sigma_1, \ldots, \sigma_n\}$. Show that $(\beta_1, \ldots, \beta_n)$ is a basis for L over K if and only if $\det(\sigma_i(\beta_j)) \neq 0$.

(8) [Jacobson, p. 243 #3] Let $E = \mathbb{Q}(r)$ where $r^3 + r^2 - 2r - 1 = 0$. Verify that $r' = r^2 - 2$ is also a root of that equation. Determine $\mathrm{Aut}(E/\mathbb{Q})$, and show that E is normal over $\mathbb{Q}$.

(9) [Jacobson, p. 243 #7] Let $E = \mathbb{Z}_p(t)$, where t is transcendental over $\mathbb{Z}_p$. Let G be the group of automorphisms generated by the automorphism of E sending $t \mapsto t + 1$. Determine $F = \mathrm{Inv}(G)$ and $[E{:}F]$.

(10) Let L/K be an extension of degree 2, that $L^2 = L$, $\mathrm{char} K \neq 2$, and that every polynomial of odd degree in $K[x]$ has a root in K. Given $f \in K[x]$, let M/L be a splitting field extension for f over L; and put $G := Aut(M/K)$, with subgroup $H := Aut(M/L)$.
(i) Show that $|G| = 2^n$. [Hint: consider the fixed field of a Sylow 2-subgroup of G.]
(ii) Show that if $n > 1$ then there is an irreducible quadratic in $L[x]$. [Hint: consider a subgroup of index 2 in H.]
(iii) Show that L is algebraically closed.
(iv) Apply your result to $\mathbb{C}/\mathbb{R}$ to obtain an algebraic proof that $\mathbb{C}$ is algebraically closed.