PROBLEM SET 6

The last exercise uses stuff we'll discuss on Wednesday (but is easy).

(1) Suppose that L/K is finite and separable and M/L is finite simple. Show that M/K is simple.

(2) [Jacobson, p. 295 #2] Find a primitive element for a splitting field over $\mathbb{Q}$ of $x^5 - 2$.

(3) [Jacobson, p. 250 #6] Define G^i by $G^1 = G$, $G^i = [G^{i-1}, G]$ (commutators). The sequence of normal subgroups $G^1 \supset G^2 \supset \cdots$ is called the *lower central series* for G. G is called *nilpotent* if there exists an integer k such that $G^k = 1$. Show that if G is nilpotent, then it is solvable. Give an example to show that the converse does not hold.

(4) [Jacobson, p. 256 #1] Let p be a prime unequal to the $\mathrm{char}(K)$.[1] Show that, if $a \in K$, then $x^p - a$ is either irreducible in $K[x]$ or it has a root in K.

(5) [Jacobson, p. 256 #2] Assume that $x^p - a$, $a \in \mathbb{Q}$, is irreducible in $\mathbb{Q}[x]$. Show that the Galois group of $x^p - a$ over $\mathbb{Q}$ is isomorphic to the group of transformations of $\mathbb{Z}_p$ of the form $y \mapsto by + c$, where $b, c \in \mathbb{Z}_p$ and $b \neq 0$.

(6) [Jacobson, p. 260 #2] Let $K = \mathbb{R}$ and let $f(x)$ be a cubic with discriminant Δ. Show that $f(x)$ has multiple roots, three distinct real roots, or one real root and two non-real roots according as $\Delta = 0$, $\Delta > 0$, or $\Delta < 0$.

(7) [Jacobson, p. 260 #7] Show that the transitive subgroups of $\mathfrak{S}_4$ are (i) $\mathfrak{S}_4$, (ii) $\mathfrak{A}_4$, (iii) $V_4 = \{\mathbf{1}, (12)(34), (13)(24), (14)(23)\}$, (iv) $C_4 = \langle (1234) \rangle$ and its conjugates, and (v) $D_4 = V_4 \cup \{(12), (34), (1423), (1324)\}$ (which is a Sylow 2-subgroup) and its conjugates.

(8) [Jacobson, p. 266 #1(a)] Solve the equation $x^3 - 2x + 4 = 0$ (in an extension of $\mathbb{Q}$) by Cardano's formulas.

[1] This assumption (made by Jacobson) is in fact unnecessary, though you may use it if you wish.