Problem Set 7

(1) Suppose that $f = x^n + px + q$. Show that $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-2} = 0$, $\lambda_{n-1} = -(n-1)p$, $\lambda_n = -nq$, $\lambda_{n+1} = \cdots = \lambda_{2n-3} = 0$, and $\lambda_{2n-2} = (n-1)p^2$. Hence show the discriminant Δ of f is $\varepsilon_n n^n q^{n-1} + \varepsilon_{n-1}(n-1)^{n-1}p^n$, where $\varepsilon_n = 1$ if $n \underset{(4)}{\equiv} 0,1$ and $\varepsilon_n = -1$ if $n \underset{(4)}{\equiv} 2,3$. [Hint: use Newton's identities on p. 140 of Jacobson.]

(2) Using our methods for dealing with quartics (and problem (1)), determine Galois groups (over $\mathbb{Q}$) of
(a) $x^4 + 8x - 12$,
(b) $x^4 + 1$,
(c) $x^4 + x^3 + x^2 + x + 1$,
(d) $x^4 - 2$.

(3) Using what we've learned about cyclotomic polynomials Φ_m, determine Galois groups (over $\mathbb{Q}$) of $x^4 + 1$ and $x^5 + 1$. [Hint: you may use Theorem 4.21 of Jacobson. I'm aware that this repeats one of the polynomials in (2).]

(4) Suppose that p is prime and doesn't divide m, and let ε be a primitive m^{th} root of 1 over $\mathbb{Z}_p$. Show that $[\mathbb{Z}_p(\varepsilon):\mathbb{Z}_p] = k$, where k is the order of $\bar{p}$ in $\mathbb{Z}_m^*$. Show that the cyclotomic polynomial Φ_m is irreducible over $\mathbb{Z}_p$ if and only if $\mathbb{Z}_m^*$ is a cyclic group generated by $\bar{p}$. When is Φ_4 irreducible over $\mathbb{Z}_p$? How about Φ_8?

(5) Determine whether Φ_{18} is irreducible over $\mathbb{Z}_{23}$, $\mathbb{Z}_{43}$, and $\mathbb{Z}_{73}$. (You may want to look at I.L.22 and the comments after it.)

(6) Show that the primitive n^{th} roots of 1 over $\mathbb{Q}$ form a normal basis for the splitting field of $x^n - 1$ over $\mathbb{Q}$ if and only if n has no repeated prime factors.

(7) [Jacobson, p. 287 #1] Show that $\sin(u)$ is transcendental for all algebraic $u \neq 0$.

(8) Suppose L/K is an extension, and that L is finitely generated over K. Show that the field K_a of elements of L which are algebraic over K is f.g. over K.