PROBLEM SET 8

(1) [Jacobson p. 100 #7] Let m and n be non-zero integers and let R be the subset of $M_2(\mathbb{C})$ consisting of the matrices of the form

$$\begin{pmatrix} a+b\sqrt{m} & c+d\sqrt{m} \\ n(c-d\sqrt{m}) & a-b\sqrt{m} \end{pmatrix}$$

where $a,b,c,d \in \mathbb{Q}$. Show that R is a subring of $M_2(\mathbb{C})$ and that R is a division ring if and only if the rational numbers x,y,z,t satisfying the equation $x^2 - my^2 - nz^2 + mnt^2 = 0$ are $x = y = z = t = 0$. Give a choice of m,n for which r is a division ring and a choice of m,n for which R is not a division ring. [N.B. These rings are called "rational quaternion algebras".]

(2) [Jacobson, p. 300 #1] Show that if E is a finite field and F is a subfield, so that E/F is a cyclic extension, then the norm homomorphism $N_{E/F}$ of E^* is surjective on F^*.

(3) In this problem you will prove a special case of the Kronecker-Weber theorem, that every abelian extension of $\mathbb{Q}$ is a subfield of a cyclotomic field. The first 3 parts come from [Jacobson, pp. 276-277].

(a) Suppose $f(x) \in K[x]$ (of degree n) has n distinct roots r_i in a splitting field. Show that the discriminant Δ is equal to $(-1)^{n(n-1)/2}\prod_{i=1}^{n} f'(r_i)$.

(b) Let p be an odd prime. By differentiating $x^p - 1 = (x-1)\Phi_p(x)$, show that the discriminant of Φ_p is $(-1)^{p(p-1)/2}p^{p-2}$.

(c) Show that $\mathbb{Q}(\zeta_p)$ has a unique subfield E with $[E{:}\mathbb{Q}] = 2$, which is real or not depending on whether p has the form $4n+1$ or $4n+3$.

(d) Prove that for each $m \in \mathbb{Z}\backslash\{0\}$, $\mathbb{Q}(\sqrt{m})$ is a subfield of $\mathbb{Q}(\zeta_{4|m|})$. [Hint: don't forget that $\mathbb{Q}(\zeta_M) \subset \mathbb{Q}(\zeta_N)$ if $M \mid N$.]