PROBLEM SET 9

(1) What are the semisimple $\mathbb{Z}$-modules?
(2) Show that a (left) R-module M has a composition series iff it is Noetherian and Artinian, cf. the remarks just before III.A.4.
(3) Show that the center of a semisimple ring R is a finite direct product of fields. Show that, in the case $R \cong M_n(D)$ (D a division ring), it is a field.
(4) Let M be a finitely generated (left) R-module and $E = \mathrm{End}_R(M)$. Show that if R is semisimple, then so is E.
(5) Let R be an n^2-dimensional algebra over a field k. Show that $R \cong M_n(k)$ (as k-algebras) if and only if R is simple[1] and has an element r whose minimal polynomial over k has the form $(x - a_1) \cdots (x - a_n)$ where $a_i \in k$. [Hint: for the "if" part, derive from the existence of the element r that ${}_R R$ has a composition series of length $\geq n$.]

[1]Here I mean simple in the weaker (k-algebra) sense, that R has no nontrivial proper 2-sided ideals.