Problem Set 1

Do problems 1 and 2. Then do 3 or 4, and 5 or 6. Use only the material we've covered so far in the notes. (You may use results stated but not proved, provided they aren't exercises!)

(1) Let M be a complex manifold. Assume M is compact and connected, and let $f \in \mathcal{O}(M)$. Show that f is constant.

(2) Let V and W be vector spaces (over the same field). Prove that $\bigwedge^n(V \oplus W) \cong \oplus_{p+q=n} \wedge^p V \otimes \wedge^q W$.

(3) In the following, I is a multi-index, x_j are local coordinates on a manifold, ω and μ are p- resp. q-forms, and ξ is a vector field. Show the following (locally is ok):

(a) $\frac{\partial}{\partial x_j} \lrcorner dx_I = \begin{cases} 0, & j \notin I \\ (-1)^{\ell-1} dx_{I \setminus \{j\}}, & j = i_\ell \in I \end{cases}$

(b) $\xi \lrcorner (\mu \wedge \omega) = (\xi \lrcorner \mu) \wedge \omega + (-1)^q \mu \wedge (\xi \lrcorner \omega)$

(4) Check the formula (cf. page 6 of the notes)

$$d\omega(\xi^0, \dots, \xi^p) = \textstyle\sum_{0 \le j \le p} (-1)^j \xi_j \{\omega(\xi^0, \dots, \widehat{\xi^j}, \dots, \xi^p)\} + \sum_{0 \le j < k \le p} (-1)^{j+k} \omega([\xi^j, \xi^k], \xi^0, \dots, \widehat{\xi^j}, \dots, \widehat{\xi^k}, \dots, \xi^p)$$

in the special case where $\dim M = 2$ and ω is a 1-form (so $p = 1$). (Just do it in local coordinates x, y.)

(5) Make explicit and verify the formula $\Pi_* d\omega + d\Pi_* \omega = I_1^* \omega - I_0^* \omega$ from the notes (p. 8).

(6) Show that the (real) de Rham cohomology of $\mathbb{R}^2/\mathbb{Z}^2$ has (in degrees $0, 1, 2$) respective dimensions $1, 2, 1$. [Hint: the torus acts transitively on itself by translation. Use the Observation on p. 8 of the notes.]