Problem Set 2

Hand in at least 1-4.

(1) Show that the wedge product on forms induces a well-defined cup-product on Dolbeault cohomology.

(2) Local $\partial\bar{\partial}$-lemma: let M be a complex manifold, ω a smooth real closed form of type $(1,1)$ on M. Show, in a neighborhood of each point, that there exists a real-valued smooth function h such that $\omega = \sqrt{-1}\partial\bar{\partial}h$. [Hint: you'll want to use the d- and $\bar{\partial}$-Poincare lemmas and also that $\partial\bar{\partial} = -\bar{\partial}\partial$.]

(3) Let M be a compact, connected smooth manifold of (real) dimension m. Prove the following:
(a) If M is orientable, then $H^m_{\mathrm{dR}}(M, \mathbb{R}) \neq \{0\}$. [Hint: use a prtition of unity and Stokes's theorem.]
(b) Assume m is even. If M has an almost complex structure, then M is orientable. [Hint: have a look at section I.A.]

(4) Let M be a smooth manifold, J an almost complex structure on M. Define (for all $X, Y \in C^\infty(M, T_M)$)

$$N(X,Y) := [JX, JY] - [X,Y] - J[JX,Y] - J[X,JY].$$

(a) Show that N is bilinear over $C^\infty(M)$ and antisymmetric. [Hint: see p. 47 of Voisin.]
(b) Prove that J is integrable if and only if $N = 0$.

(5) Show that the almost complex structure in Example I.D.1 (pp. 24-25) on S^6 is not integrable.