Problem set 3

Do 1 or 4, and do 2, 3, and 5.

(1) (i) Let $E \to M$ be a holomorphic vector bundle over a complex manifold. Using a partition of unity, show that a Hermitian metric h on E always exists.
(ii) Now suppose that $E =: L$ is a line bundle, and define (from h) on coordinate neighborhoods $\mathcal{U}_\alpha$ the functions ρ_α as on p. 43 of the notes. Show that the $1/\rho_\alpha$ correspond to a Hermitian metric on $L^\vee$ (call this h^*), and conclude that $c_1(L^\vee) = -c_1(L)$.

(2) (i) Prove that the tautological bundle $\mathcal{O}(-1) \to \mathbb{P}^n$ has no non-trivial global holomorphic sections.
(ii) Show that the canonical bundle $K_{\mathbb{P}^n}$ is isomorphic to $\mathcal{O}(-n-1)$. You will need to first construct a trivialization (using a section over each U_i), and compute transition functions.

(3) (i) Let M be a complex manifold with submanifold N of codimension 1, with $\mathcal{N}_{N/M}$ the normal bundle. Prove the adjunction formula $K_N \cong K_M|_N \otimes \mathcal{N}_{N/M}$.
(ii) Let $X = \bar{V}(F)$, $F \in S^d_{n+1}$, be a smooth hypersurface of degree d in $\mathbb{P}^n$. Write $\mathcal{O}_X(m)$ for the restriction $\mathcal{O}_{\mathbb{P}^n}(m)$. Show that $\mathcal{N}_{X/\mathbb{P}^n} \cong \mathcal{O}_X(d)$; compute K_X in the same terms.
(iii) Prove that for $d = n+1$, there exists a global nowhere-vanishing holomorphic form $\omega \in \Omega^{n-1}(X)$, and that $\dim_{\mathbb{C}}(\Omega^{n-1}(X)) = 1$.

(4) Show that the vanishing of the torsion of a connection is equivalent to $[\chi, \xi] = \nabla_\chi \xi - \nabla_\xi \chi$ for any two vector fields.

(5) Write local coordinate systems and transition functions for (a) the blowup B_0 of $\mathbb{C}^2$ at $(0,0)$ and (b) the blowup B_1 of B_0 at some point on the preimage of $(0,0)$. So we have maps $B_1 \to B_0 \to \mathbb{C}^2$. Compute the pullback (in each of your neighborhoods) of $dz_1 \wedge dz_2 \in \Omega^2(\mathbb{C}^2)$ to each. How does it behave on the preimage of $(0,0)$ – does it have zeroes or poles, and of what order?