Problem set 4

(1) Referring to the diagram for the five-lemma on p. 66 of the notes, you'll recall that I did injectivity of f_3 in class. Prove the surjectivity of f_3.

(2) This problem is about the covariant functor $Hom_{\mathcal{C}}(X, -)$ where $X \in \mathcal{C}$, and its first right-derived functor $Ext^1_{\mathcal{C}}(X, -)$.
(a) Show in general that $Hom(X, -)$ is left-exact.
(b) Specializing to $\mathcal{C} = Ab$, find an injective resolution of $\mathbb{Z}_m$.
(c) Use this resolution to compute $Ext^1(\mathbb{Z}_m, \mathbb{Z}_n)$.

(3) This problem is set on a compact Riemann surface. Let $\mathcal{M}^*$ be the sheaf of not-identically-zero meromorphic functions, and $\mathcal{D}$ the sheaf of divisors (so that $H^0(\mathcal{D}) = Div(M)$). Show that every line bundle $L \to M$ is of the form $\mathcal{O}(D)$ for some $D \in Div(M)$ if and only if $H^1(M, \mathcal{M}^*) = \{0\}$. [Hint: start by writing down a short exact sequence involving $\mathcal{D}$ and $\mathcal{M}^*$, and checking that $\mathcal{D}$ is flasque (or fine).]

(4) Let M be a compact Riemann surface of genus g, $D \in Div(M)$.
(a) Prove that if $\deg D > 2g - 2$, then $i(D) = 0$.
(b) Likewise, show that if $\deg D < 0$, then $\ell(D) = 0$.

(5) Next let M have genus $g \geq 2$.
(a) Prove that M has a morphism (i.e. holomorphic map) to $\mathbb{P}^1$ of degree $\leq g + 1$. [Hint: try thinking about functions with pole at one point, and otherwise holomorphic.]
(b) Prove that M has a morphism to $\mathbb{P}^1$ of degree $\leq g$. [Hint: let $p \in M$, and look at $i((g-2)[p])$.]