

PROBLEM SET 5

Page numbers refer to my notes. In addition to these exercises, you should look at the two exercises at the end of Chapter 6 in Voisin. (If you're feeling particularly adventurous, you could also try some of those at the end of Chapter 6 in Warner.)

- (1) Verify that $\partial/\partial\bar{z}$ is an elliptic operator on \mathbb{C} -valued functions on $U \subset \mathbb{C}$.
- (2) (i) Check that Δ_d commutes with Hodge $*$. (ii) Given $\alpha \in A^r(M)$ (M compact Riemannian manifold) closed, show that $*\alpha$ is closed if and only if α is harmonic. (iii) If M is complex Hermitian, prove that $*$ takes forms of bidegree (p, q) to forms of bidegree $(n - q, n - p)$.
- (3) Consider the Heisenberg group $M = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{C} \right\} \subset GL_3(\mathbb{C})$ and put $\Gamma := M \cap GL_3(\mathbb{Z})$. Show that the Iwasawa manifold $\Gamma \backslash M$ is non-Kähler by exhibiting a non-closed holomorphic form. [Hint: consider " $M^{-1}dM$ ".]
- (4) Let $L \rightarrow X$ be a holomorphic line bundle over a compact complex n -manifold, such that for some $N > 0$ $H^0(X, \mathcal{O}(L^{\otimes N})) \neq 0$. Prove that if also $H^n(X, K_X \otimes \mathcal{O}(L)) \neq 0$, then L is trivial.
- (5) Prove the pairings Q_k (cf. p. 156) are well-defined.
- (6) Check the well-definedness of Poincare residue (i.e. independence of choice of local coordinates).
- (7) Compute $h^{1,1}$ for quintic surfaces in \mathbb{P}^3 and $h^{2,1}$ for quintic threefolds in \mathbb{P}^4 .
- (8) Work out the constants in powers of Laurent polynomials and check the Picard-Fuchs equation (F.11) on pp. 175-6, in the special case $n = 2$. This is a degree 3 ODE; in fact one can do better: find a degree 2 differential equation satisfied by $\mathfrak{P}(t)$. Then find the first few terms of a solution of the form $\log(t)\mathfrak{P}(t) + \mathfrak{Q}(t)$ (where \mathfrak{Q} is a power series about $t = 0$ like \mathfrak{P}).