Problem set 6

Page numbers refer to my notes. There are more problems in the notes; I've picked the most important ones.

(1) Let $(V_{\mathbb{Z}}, F^\bullet)$ be a Hodge structure of weight n, and $U_{\mathbb{Z}} \subseteq V_{\mathbb{Z}}$ a subgroup. Prove that, together with this, the filtration $U_{\mathbb{C}}$ yields a (sub-)HS $\iff U_{\mathbb{C}} = \oplus_{p+q=n} V^{p,q} \cap U_{\mathbb{C}}$. Show that this holds, in particular, for the kernel of a morphism of HS.

(2) Let $V_{\mathbb{Z}}$ be an abelian group of even rank (OK to assume torsion-free), Q an alternating nondegenerate bilinear form on $V_{\mathbb{Z}}$, $n \in \mathbb{N}$ odd, $\underline{h} = (h^{n,0}, h^{n-1,1}, \ldots, h^{0,n})$, and D the associated classifying space for HS on $V_{\mathbb{Z}}$ of weight n and type $\underline{h}$ polarized by Q. Show that the real symplectic group $G_{\mathbb{R}} := Aut(V_{\mathbb{R}}, Q)$ acts transitively on D. Don't try to do it in general at first: start with $n = 1$. [Hint: every HS in D has a basis $\{\omega_j\}$ subordinate to the Hodge decomposition and closed under complex conjugation, which can be taken to satisfy $Q(\omega_i, \bar{\omega}_j) = \xi_i \delta_{ij}$ (where $\xi_i = \pm\sqrt{-1}$). Consider the transformation moving one such basis to another (for a different PHS in D); is it in $G_{\mathbb{R}}$?]

(3) [to do after (2)] Prove in the setting of problem (2) that the isotropy subgroup $H \subset G_{\mathbb{R}}$ fixing a reference Hodge structure is in fact the product of unitary groups claimed on p. 182.

(4) Show that the inclusion of complexes $\Omega_M^{\bullet \geq p} \hookrightarrow F^p A_M^\bullet$ is a quasi-isomorphism. [Hint: use $\bar{\partial}$-Poincaré and (the total complex of) a double complex of sheaves.]

(5) Use the long-exact sequences associated to the two short-exact sequences on p. 189 to define a map $d_1 : H^{p+q}(Gr_F^p \mathcal{C}^\bullet) \to H^{p+q+1}(Gr_F^{p+1} \mathcal{C}^\bullet)$, and check that $d_1 \circ d_1 = 0$.

(6) Show that the infinitesimal period relation is trivial on the period domain for PHS of weight $n = 2$ and Hodge numbers $h^{2,0} (= h^{0,2}) = 1$ ($h^{1,1}$ arbitrary). (You don't have to worry about the precise integral form of Q, just that HR I/II hold.) [Hint: compute the tangent space to D and show, using the infinitesimal form of HR I for Q (cf. p. 205), that it is already contained in $Hom(F^2, F^1/F^2) \oplus Hom(F^1, V/F^1)$.]

(7) Derive Example III.C.1 (p. 201) from the Picard-Fuchs equation in Problem Set 5 Exercise 8. (The function $g(s)$ should be related to $\mathfrak{Q}/\mathfrak{P}$ in that problem.) Also, show how the "renormalized" version with parameter q arises from the family of complex 1-tori $\mathbb{C}/\mathbb{Z}\langle 1, \tau \rangle$ over the upper half plane.