Hodge Theory Midterm Solutions

1) a) We are, according to Theorem I.G-3, allowed to use complex $\text{“}R\text{”}$ row operations & integral $\text{“}C\text{”}$ column operations to try to put the lattice/period-matrix in the form $(I \;\; Z)$ with ${}^tZ = Z$ & $\operatorname{Im}(Z) > 0$.

note: $\zeta = e^{\frac{2\pi i}{5}}$

We have
$$\begin{pmatrix} 1 & \zeta & \zeta^2 & \zeta^3 \\ 1 & \zeta^2 & \zeta^4 & \zeta \end{pmatrix} \xrightarrow{C} \begin{pmatrix} 1 & \zeta^2 & \zeta & \zeta^3 \\ 1 & \zeta^4 & \zeta^2 & \zeta \end{pmatrix} \xrightarrow{R} \begin{pmatrix} 1 & \zeta^2 & \zeta & \zeta^3 \\ 0 & \zeta^4-\zeta^2 & \zeta^2-\zeta & \zeta-\zeta^3 \end{pmatrix}$$
$$\xrightarrow{R} \begin{pmatrix} 1 & \zeta^2 & \zeta & \zeta^3 \\ 0 & \zeta^2 & (-\zeta^3-\zeta^4) & -\zeta \end{pmatrix} \xrightarrow{R} \begin{pmatrix} 1 & 0 & \zeta+\zeta^2+\zeta^4 & \zeta^3+\zeta \\ 0 & \zeta^2 & -\zeta^2-\zeta^4 & -\zeta \end{pmatrix} \xrightarrow{R} \begin{pmatrix} 1 & 0 & -(1+\zeta^3) & \zeta^3+\zeta \\ 0 & 1 & -(1+\zeta^2) & -\zeta^4 \end{pmatrix}$$

divide 2nd row by ζ^2-1 & use $1+\zeta+\zeta^2+\zeta^3+\zeta^4=0$

$$\xrightarrow{C} \begin{pmatrix} 1 & 0 & \boxed{\begin{matrix} \zeta-1 & \zeta^3+\zeta \\ \zeta+\zeta^3 & -\zeta^4 \end{matrix}} \\ 0 & 1 & \end{pmatrix}$$
Z where ${}^tZ = Z$. Moreover,

$$\operatorname{Im} Z = \frac{1}{i}\begin{pmatrix} \zeta-\zeta^4 & \zeta-\zeta^2+\zeta^3-\zeta^4 \\ \zeta-\zeta^2+\zeta^3-\zeta^4 & -\zeta^4+\zeta \end{pmatrix} = \begin{pmatrix} A & a \\ a & A \end{pmatrix} \text{ where } A > a > 0 \implies \text{positive definite.}$$

So the first one is an abelian variety.

The second is not, as $\sqrt{-2}, \sqrt{-3}, \sqrt{-5}, \sqrt{-7}$ are algebraically independent and $\therefore$ no column or row operations can change the situation ${}^tZ \neq Z$.

b) The nontrivial automorphism is given by

$$\begin{aligned} \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ (z_1, z_2) &\longmapsto (\zeta z_1, \zeta^2 z_2) \end{aligned} \quad ;$$

of course, what we must show is that this gives an automorphism of the lattice: it sends Λ to

$$\hookrightarrow \mathbb{Z}\left\langle \begin{pmatrix} \zeta \\ \zeta^2 \end{pmatrix}, \begin{pmatrix} \zeta^2 \\ \zeta^4 \end{pmatrix}, \begin{pmatrix} \zeta^3 \\ \zeta \end{pmatrix}, \begin{pmatrix} \zeta^4 \\ \zeta^3 \end{pmatrix} \right\rangle = \mathbb{Z}\left\langle \begin{pmatrix} \zeta \\ \zeta^2 \end{pmatrix}, \begin{pmatrix} \zeta^2 \\ \zeta^4 \end{pmatrix}, \begin{pmatrix} \zeta^3 \\ \zeta \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$$

Since $\begin{pmatrix} \zeta^4 \\ \zeta^3 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \zeta \\ \zeta^2 \end{pmatrix} - \begin{pmatrix} \zeta^2 \\ \zeta^4 \end{pmatrix} - \begin{pmatrix} \zeta^3 \\ \zeta \end{pmatrix}$

[using that $1+\zeta+\zeta^2+\zeta^3+\zeta^4=0$].

By pullback, this induces a nontrivial automorphism of the HS on $H^1(T_1)$. ~~Indeed~~, it is given by a $\mathbb{Z}$-matrix on the $\mathbb{Z}$-basis and a diagonal matrix on the Hodge basis.

$\in \mathbb{R}$

2) Let $\xi = \boxed{\alpha}\, dx + \boxed{\beta}\, dy + \boxed{\gamma}\, dz + \boxed{\delta}\, dw \in T_p^\vee \mathbb{R}^4 \setminus \{0\}$, $p \in U$,

and consider the $\mathbb{H}$-valued functions as sections of the trivial bundle $\underset{(\cong \mathbb{H})}{\mathbb{R}^4} \times U \to U$. Multiplication by $1, i, j, k$ have matrix representations (in the basis $\{1, i, j, k\}$ of $\mathbb{H}$ over $\mathbb{R}$)

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, \quad \begin{pmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}, \quad \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \text{ (resp.)}$$

Hence,
$$\sigma(D)(\xi) = \sigma\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} - j\frac{\partial}{\partial z} - k\frac{\partial}{\partial w}\right)(\alpha\, dx + \beta\, dy + \gamma\, dz + \delta\, dw)$$

$$= \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \alpha & \\ & & & \alpha \end{pmatrix} - \begin{pmatrix} & -\beta & & \\ \beta & & & \\ & & & -\beta \\ & & \beta & \end{pmatrix} - \begin{pmatrix} & & -\gamma & \\ & & & \gamma \\ \gamma & & & \\ & -\gamma & & \end{pmatrix} - \begin{pmatrix} & & & -\delta \\ & & -\delta & \\ & \delta & & \\ \delta & & & \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & +\beta & +\gamma & +\delta \\ -\beta & \alpha & +\delta & -\gamma \\ -\gamma & -\delta & \alpha & +\beta \\ -\delta & +\gamma & -\beta & \alpha \end{pmatrix}.$$

We need to show that ($\xi \neq 0 \Rightarrow$) this is injective as a map from $\mathbb{R}^4 \to \mathbb{R}^4$, i.e. that its determinant is nonvanishing. A short computation reveals that this determinant is $(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^2$ $(\neq 0)$.

You could also be a bit more clever about this by using the norm on $\mathbb{H}$ instead of computing the determinant. After all, $\underline{\mathbb{H} \text{ is a division ring}}$.

3) From p. 173, Prop. 1 we already knew that

$H^{1,1}_{pr}(X)$ has dimension $h^{1,1}(X) - 1 = 45 - 1 = 44$.

The basis can be given via the Poincaré residue isomorphism (noting $kd - n - 1 = 2\cdot 5 - 3 - 1 = 6$)

$$\begin{tikzcd} R_F^6 := S^6/J_F^6 \arrow[r, "\cong"] & Gr_2^{\mathcal{F}} H^3(X) \arrow[r, "\cong"] & H^{1,1}_{pr}(X,\mathbb{C}) \\ P \arrow[r, mapsto] & \frac{P\Omega}{F^2} \arrow[r, mapsto] & \mathrm{Res}_X\left(\underbrace{\left[\frac{P\Omega}{P^2} + d\xi\right]}_{\text{dlog form}}\right) \end{tikzcd}$$

$F := z_0^5 + z_1^5 + z_2^5 + z_3^5$ for ~~Fermat~~ quintic $\mathbb{CP}^3$

So really all we need is an explicit monomial basis for $S^6/(z_0^4, z_1^4, z_2^4, z_3^4)^6$, which consists of the

- $z_i^3 z_j^3$ —— $\binom{4}{2}$ of them
- $z_i^3 z_j^2 z_k$ —— $4!$ ...
- $z_0^3 z_j z_k z_\ell$ —— $\binom{4}{1}$ ...
- $z_i^2 z_j^2 z_k z_\ell$ —— $\binom{4}{2}$ ...
- $z_i^2 z_j^2 z_k^2$ —— $\binom{4}{3}$ ...

total $= 6 + 24 + 4 + 6 + 4 = 44$.

4) a) The HS on $H^1(E)$ $E = \mathbb{C}/\mathbb{Z}\langle 1, \tau \rangle$, has period matrix

$$\begin{array}{c|cc} & \beta & \alpha \\ \hline \omega : & 1 & \tau \\ \bar{\omega} : & 1 & \bar{\tau} \end{array}$$

where ω generates $H^{1,0}(E)$ & α, β generate $H^1(E, \mathbb{Z})$.
[This means: $\omega = \beta + \tau\alpha$]

So for the tensor product of this HS with itself, which is of weight 2, we have

$$\begin{array}{c|cccc} & \beta\otimes\beta & \beta\otimes\alpha & \alpha\otimes\beta & \alpha\otimes\alpha \\ \hline \omega\otimes\omega & 1 & \tau & \tau & \tau^2 \\ \omega\otimes\bar{\omega} & 1 & \bar{\tau} & \tau & \tau\bar{\tau} \\ \bar{\omega}\otimes\omega & 1 & \tau & \bar{\tau} & \tau\bar{\tau} \\ \bar{\omega}\otimes\bar{\omega} & 1 & \bar{\tau} & \bar{\tau} & \bar{\tau}^2 \end{array}.$$

$\leftarrow$ in fact, this is enough to specify the HS if you know the polarization Q, because it gives F^2 and then $F^1 = (F^2)^{\perp}$ by HRBR I.

b) $\sigma : E \times E \to E \times E$
$(p, q) \mapsto (q, p)$

induces (by pullback) a morphism of HS from $H^1(E) \otimes H^1(E)$ to itself (swapping the factors).

Since $\sigma^2 = \text{id.}$, there are 2 possible eigenvalues: ± 1.

Moreover, $\sigma + \text{id}$ is a morphism of HS, and so its kernel is a sub HS of $H^1(E) \otimes H^1(E)$ (if it is nonzero). It is easy to check that, indeed, the kernel is generated by $\omega\otimes\bar{\omega} - \bar{\omega}\otimes\omega$ (or by $\beta\otimes\alpha - \alpha\otimes\beta$). So there is a rank 1 sub HS of pure type (1,1). Indeed, as HS we have a splitting $\boxed{H^1(E)\otimes H^1(E) \cong \ker(\sigma + \text{id.}) \oplus \ker(\sigma - \text{id.})}$.

5) a) We are in weight 2, so Q is symmetric. Let V_0 be the reference HS. Since $V_0^{2,0}$ is 1-dimensional, if $u \in V_0^{2,0}\setminus\{0\}$ and $v \in V_0^{0,2}\setminus\{0\}$, then by HR II we have $Q(v,u) \neq 0$. (*)

Now recall that, writing

$$\mathfrak{g} = \{X \in \mathfrak{gl}(V_{0,\mathbb{C}}) \mid Q(Xu,v)+Q(u,Xv)=0 \ (\forall u,v \in V_{0,\mathbb{C}})\},$$

$T_0^{1,0}D \cong \bigoplus_{i>0} \mathfrak{g}^{-i,i}$ and the IPR is trivial $\Leftrightarrow$ $T_0^{1,0}D = \mathfrak{g}^{-1,1}$.

So it will suffice to check that

$$\mathfrak{g}^{-2,2} = \{X \in \mathfrak{g} \mid X(V_0^{2,0}) \subset V_0^{0,2}\} = \{0\}.$$

But given $X \in \mathfrak{g}^{-2,2}$, $u \in V_0^{2,0}\setminus\{0\}$ we have $Xu \in V_0^{0,2}$ and

$$0 = Q(Xu,u)+Q(u,Xu) \underset{\text{symm.}}{=} 2Q(Xu,u) \underset{(*)}{\Longrightarrow} Xu = 0.$$

Hence $X=0$, done.

b) Consider 2 HS's V_0, $\tilde{V}_0$ on V polarized by Q, both of weight 2 & w/ Hodge #'s $h^{2,0}=1=h^{0,2}$, $h^{1,1}=a$. By the Hodge-Riemann bilinear relations + linear algebra, we may choose

$$\begin{cases} \omega_0 \in V_0^{2,0}, & \tilde{\omega}_0 \in \tilde{V}_0^{2,0} \\ \eta_1,\dots,\eta_a \in V_{0,\mathbb{R}}^{1,1} & \tilde{\eta}_1,\dots,\tilde{\eta}_a \in \tilde{V}_{0,\mathbb{R}}^{1,1} \end{cases}$$

s.t.

$$\begin{cases} Q(\eta_i,\eta_j) = \delta_{ij} = Q(\tilde{\eta}_i,\tilde{\eta}_j) \\ -Q(\omega_0,\bar{\omega}_0) = 1 = -Q(\tilde{\omega}_0,\bar{\tilde{\omega}}_0). \end{cases}$$

b/c by (*) it commutes with complex conjugation.

Now consider the linear automorphism of $V_{\mathbb{C}}$ sending

$$(*)\begin{cases} \omega_0 \mapsto \tilde{\omega}_0, & \bar{\omega}_0 \mapsto \bar{\tilde{\omega}}_0 \\ \eta_j \mapsto \tilde{\eta}_j & (\forall j) \end{cases}$$

This is clearly defined over $\mathbb{R}$ & also clearly preserves Q. Hence, it belongs to $G_{\mathbb{R}} = \mathrm{Aut}(V_{\mathbb{R}}, Q)$. We conclude that $G_{\mathbb{R}}$ acts transitively on D.

6) We want to compute d_2 in the spectral sequence of the double complex

$$\begin{tikzcd}
 & & \\
A^k(D^{[1]}) \arrow[u, "d"] \arrow[r, "\delta"] \arrow[dr, dashed] & A^k(D^{[2]}) \arrow[u] \arrow[r, "\delta"] & A^k(D^{[3]}) \arrow[u] \\
A^{k-1}(D^{[1]}) \arrow[u, "d"] \arrow[r, "\delta"] & A^{k-1}(D^{[2]}) \arrow[u, "d"] \arrow[r] \arrow[r, dashed, bend left, "\delta"] & A^{k-1}(D^{[3]}) \arrow[u, "d"] \\
 \arrow[u] & \arrow[u] & \arrow[u]
\end{tikzcd}$$

The idea is to "follow the dotted arrow"; when $\xi \in A^k(D^{[1]})$ is in $\ker(d)$ (hence survives to E_1) and then in $\ker([\delta])$ (hence surviving to E_2), this makes sense. The key point is that given $\xi \in F^p A^k(D^{[1]}) \cap \ker(d) \cap \ker([\delta])$, we can choose the form $\eta \in A^{k-1}(D^{[1]})$ with $d\eta = \delta\xi$ to be in $F^p A^{k-1}(D^{[1]})$ (by Corollary II.D.1(b)). So d_2 preserves $F^\bullet$.

Furthermore, d_2 is "defined over $\mathbb{R}$", because we can consider differential forms with real coeffs. (and all the maps here are the complex linear extensions of these). Consequently, d_2 preserves $\overline{F}^\bullet$ too.

Now, d_2 goes from a sub HS of $H^k(D^{[1]})$ to a subquotient HS of $H^{k-1}(D^{[3]})$. Given $\xi \in H^{p,q}(D^{[1]})$ $(p+q=k)$ $= (F^p \cap \overline{F}^q) H^k(D^{[1]})$, this is sent to $F^p \cap \overline{F}^q$ of a weight $\underline{k-1}$ HS. By the p-opposed condition, this is zero.