HODGE THEORY MIDTERM EXAM

Hand in five.

(1) Consider the two complex 2-tori $T_1 = \mathbb{C}^2/\Lambda_1$, $T_2 = \mathbb{C}^2/\Lambda_2$, where (writing ζ for a 5th root of 1)

$$\Lambda_1 = \mathbb{Z}\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \zeta \\ \zeta^2 \end{pmatrix}, \begin{pmatrix} \zeta^2 \\ \zeta^4 \end{pmatrix}, \begin{pmatrix} \zeta^3 \\ \zeta \end{pmatrix} \right\rangle$$

and

$$\Lambda_2 = \mathbb{Z}\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \sqrt{-2} \\ \sqrt{-5} \end{pmatrix}, \begin{pmatrix} \sqrt{-3} \\ \sqrt{-7} \end{pmatrix} \right\rangle.$$

(a) Decide whether each torus is an abelian variety.
(b) Find a nontrivial automorphism of T_1, and interpret this Hodge-theoretically ($H^1(T_1)$).

(2) Let $\mathbb{H}$ be Hamilton's quaternions, $U \subset \mathbb{R}^4$ be an open set, and $\mathfrak{F}$ be the set of smooth $\mathbb{H}$-valued functions on U. Show that $D := \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} - j\frac{\partial}{\partial z} - k\frac{\partial}{\partial w}$, viewed as an operator from $\mathfrak{F}$ to itself, is elliptic.

(3) Find an explicit basis for $H^{1,1}_{pr}(X)$, where X is the Fermat quintic surface in $\mathbb{P}^3$.

(4) (a) Write a period matrix for the HS on $H^1(E) \otimes H^1(E)$, where E is an elliptic curve with holomorphic differential ω.
(b) Is this HS irreducible? [Hint: consider the automorphism σ of $E \times E$ exchanging factors.]

(5) Fix $V_{\mathbb{Z}}$ and Q (symmetric), and let D be the period domain for weight 2 HS on V polarized by Q with Hodge numbers $h^{2,0} = 1 = h^{0,2}$ (so $h^{1,1} = \dim(V) - 2$).
(a) Prove the IPR is trivial. [Hint: the formula for $\mathfrak{g}$ on p. 205 may be helpful.]
(b) Prove that $G_{\mathbb{R}} = Aut(V_{\mathbb{R}}, Q)$ acts transitively on D. [See hints on HW 6.]

(6) The spectral sequence on pp. 226-7, for the double complex $K^{a,b} := A^b(D^{[a+1]})$, degenerates at E_2. Prove the weaker statement that $d_2 = 0$. [Hint: you will need to use a result in §II.D, the explicit description of d_2, and find an argument that the spectral sequence is defined over $\mathbb{Q}$ (or at least $\mathbb{R}$).]