I. Kähler geometry and sheaf cohomology

We'll build up to Kähler manifolds gradually — going from differentiable → almost complex → complex → Hermitian → Kähler.

A. Linear algebra

all vector spaces are finite dimensional

Begin with 2 vector spaces over a field $\mathbb{F}$:

$$V = \mathbb{F}\langle \{e_i\}_{i=1}^{m} \rangle, \quad m = \dim_{\mathbb{F}} V, \qquad W = \mathbb{F}\langle \{f_j\} \rangle$$

I'm not giving details of these constructions — just make sure you're familiar with them

$$V \oplus W = \mathbb{F}\langle \{e_i\}, \{f_j\} \rangle$$

$$V \otimes W = \mathbb{F}\langle \{e_i \otimes f_j\} \rangle, \qquad V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$$

$$V^{\vee} = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F}) = \mathbb{F}\langle \{e_i^{\vee}\} \rangle, \text{ where } e_i^{\vee}(e_j) := \delta_{ij}$$

- given $T : V \to V$ with matrix $[T]_e = M$, we have
  - $\longrightarrow T^* : V^{\vee} \to V^{\vee}$ (pullback of fcns.) with $[T^*]_{e^{\vee}} = {}^tM$
  - $\longrightarrow T^{\vee} : V^{\vee} \to V^{\vee}$ (s.t. $V \otimes V^{\vee} \xrightarrow{T \otimes T^{\vee}} V \otimes V^{\vee}$, both $\to \mathbb{C}$) with $[T^{\vee}]_{e^{\vee}} = {}^tM^{-1}$

$$V_{\mathbb{F}'} = \mathbb{F}'\langle \{e_i\} \rangle = V \otimes_{\mathbb{F}} \mathbb{F}' \quad \text{for } \mathbb{F}' \supseteq \mathbb{F} \text{ extension}$$

$$V^{\mathbb{F}''} = \mathbb{F}''\langle \{\alpha_k \otimes e_i\} \rangle \quad \text{for } \mathbb{F}'' \subseteq \mathbb{F} = \mathbb{F}''\langle \{\alpha_k\} \rangle$$

$$v_1 \wedge \cdots \wedge v_n \in \Lambda^n V = \mathbb{F}\langle \{e_{i_1} \wedge \cdots \wedge e_{i_n} \mid i_1 < \cdots < i_n\} \rangle = V^{\otimes n} \Big/ \langle v_1 \otimes \cdots \otimes v_n - \mathrm{sgn}(\sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \rangle$$

all relations

note "$x \wedge x = 0$"

$\dim = \binom{m}{n} \Rightarrow \Lambda^m V \cong \mathbb{F}$, $\Lambda^m T = \det(T)$

$$\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \in V^{\otimes n}$$

$\dim = \binom{m+n-1}{n}$ (box-bar constr.)

$$\sum v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

$$v_1 \cdot \cdots \cdot v_n \in \mathrm{Sym}^n V = \mathbb{F}\langle \{e_{i_1} \cdot \cdots \cdot e_{i_n} \mid i_1 \le \cdots \le i_n\} \rangle = V^{\otimes n} \Big/ \langle v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \rangle$$

We have the operations of wedge

$$\wedge : \Lambda^p V \otimes \Lambda^q V \to \Lambda^{p+q} V \quad (\text{not an } \cong)$$

$$(v_1 \wedge \cdots \wedge v_p) \otimes (v_1' \wedge \cdots \wedge v_q') \longmapsto v_1 \wedge \cdots \wedge v_p \wedge v_1' \wedge \cdots \wedge v_q'$$

— could replace all V's by $V^{\vee}$, of course and contraction / interior-product / left-hook

$$\lrcorner : \Lambda^p V \otimes \Lambda^{p+q} V^\vee \longrightarrow \Lambda^q V^\vee \quad \left(\cong (\Lambda^q V)^\vee \right)$$

$$x \otimes \alpha \longmapsto x \lrcorner \alpha, \quad \text{where } (x \lrcorner \alpha)(\underset{\in \Lambda^q V}{z}) = \alpha(z \wedge x)$$

Ex / (A.1)
$$\boxed{\bigoplus_{p+q=n} \Lambda^p V \otimes \Lambda^q W \xrightarrow[(\wedge)]{\cong} \Lambda^n (V \oplus W)}.$$

(Key formula related to decomposition into (p,q) types.)

(i) Let $V = \mathbb{C}$-vector space

"W over $\mathbb{R}$", i.e. "W an $\mathbb{R}$-vector space"

(A.2)
$$\boxed{V = W_{\mathbb{C}} \text{ for } W/\mathbb{R} \iff \exists \text{ conjugate-linear } \rho : V \to V \text{ with } \rho^2 = \mathrm{id}_V}$$

[Proof: $(\Rightarrow)$: $V_{\mathbb{C}} = W \oplus iW$, $\rho : i \mapsto -i$

$(\Leftarrow)$: $\rho \in \mathrm{End}_{(\mathbb{R})}(V^{\mathbb{R}})$, and $V^{\mathbb{R}} = \underbrace{\ker(\rho - \mathrm{id})}_{=:W} \oplus \ker(\rho + \mathrm{id})$, $\cdot i$, $\overset{\cong}{\cdot i}$.]

In this case, $\underset{W^\vee \otimes W}{\mathrm{End}(W)} \hookrightarrow \mathrm{End}(W)_{\mathbb{C}} \cong \mathrm{End}(V)$, and

important

(A.3)
$$\boxed{T \in \mathrm{End}(V) \text{ belongs to } \mathrm{End}(W) \iff \rho \circ T = T \circ \rho .}$$

(ii) Let $W = \mathbb{R}$-vector space

"complex structure" on W

(A.4)
$$\boxed{W = V^{\mathbb{R}} \text{ for } V/\mathbb{C} \iff \exists\ \mathbb{R}\text{-linear } J : W \to W \text{ with } J^2 = -\mathrm{id}_W}$$

[Proof: $(\Rightarrow)$: J = mult. by i on V ; $(\Leftarrow)$: clear.]

In this case, $W = \mathbb{R}\langle e_1, \dots, e_m, J(e_1), \dots, J(e_m) \rangle$ e'

$W_{\mathbb{C}} = \mathbb{C}\langle \underbrace{e_1 - iJ(e_1), \dots, e_m - iJ(e_m)}_{=:W_+}, \underbrace{e_1 + iJ(e_1), \dots, e_m + iJ(e_m)}_{=:W_-} \rangle$ e''

and
$$\begin{cases} [J]_{e'} = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \\ [J]_{e''} = \begin{pmatrix} iI_m & 0 \\ 0 & -iI_m \end{pmatrix} \end{cases}.$$

Moreover $\mathrm{End}(V) \hookrightarrow \mathrm{End}(W)$ and

(A.5)
$$\boxed{T \in \mathrm{End}(W) \text{ comes from } \mathrm{End}(V) \iff J \circ T = T \circ J}$$

For such a T, $[T]_{e'} = \underset{\text{symplectic}}{\begin{pmatrix} A & -B \\ B & A \end{pmatrix}}$ and $[T]_{e''} = \begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix}$.