B. Calculus on manifolds  let $k \in \mathbb{N} \cup \{\infty\}$

Definition 1: (i) A differentiable manifold $M$ (of real dim. $m$ & class $C^k$) is a (separable + Hausdorff) topological space with

("atlas")
- open cover $\{U_\alpha\}$
- homeomorphisms $\phi_\alpha : U_\alpha \to V_\alpha \overset{\text{open}}{\subseteq} \mathbb{R}^m$ ("charts")

  s.t. $\phi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(\underbrace{U_{\alpha\beta}}_{U_\alpha \cap U_\beta}) \to \phi_\alpha(U_{\alpha\beta})$ ("transition functions")

  is a $C^k$ diffeomorphism (from an open $\subset V_\beta$ to open $\subset V_\alpha$)

(ii) For $\mathcal{U} \subset M$ open* & $0 \leq s \leq k$,

$$C^s(\mathcal{U}, \mathbb{R}) := \{\text{functions } f : \mathcal{U} \to \mathbb{R} \mid f \circ \phi_\alpha^{-1} \in C^s(\phi_\alpha(U_\alpha \cap \mathcal{U}))\ (\forall \alpha)\}$$

(iii) Morphisms $C^s(M,N)$ between manifolds defined in same vein

(iv) $M$ is $\left\{\begin{array}{l}\text{orientable}\\ \text{oriented}\end{array}\right. \iff \left\{\begin{array}{l}\exists\\ \text{equipped w/}\end{array}\right.$ atlas with all $\det(\underbrace{(\phi_{\alpha\beta})_*}_{\text{Jacobian matrix}}) > 0$

A derivation at $p \in M$ is any linear functional $D : C^k(M, \mathbb{R}) \to \mathbb{R}$ satisfying the "Leibniz rule" $D(fg) = f(p)D(g) + g(p)D(f)$; let $\mathrm{Der}_p(M)$ denote the set of these — it is a vector space $/\mathbb{R}$. Moreover, if $V \ni p$ is a small nbhd., and $f_1|_V \equiv f_2|_V$, taking $g|_{M \setminus V} \equiv 1$ and (on a smaller nbhd.) $g|_{V'} \equiv 0$,

$$D(f_1 - f_2) = D(g \cdot (f_1 - f_2)) = D(g)\overset{0}{\cancel{(f_1-f_2)(p)}} + (D(f_1) - D(f_2))\overset{0}{\cancel{g(p)}} \overset{!}{=} 0$$

also $D(1) = D(1 \cdot 1) = 2(1 \cdot D(1)) \longrightarrow D(1) = 0$.

Writing $\phi_\alpha(x) = (x_1^\alpha, \ldots, x_m^\alpha)$, $\phi_\alpha(p) = (0, \ldots, 0)$, note that $\underbrace{D \circ \phi_\alpha^*}_{\tilde{D}} \in \mathrm{Der}_0(\mathbb{R}^m)$. Any $g \in C^k(\mathbb{R}^m, \mathbb{R})$ satisfies

$$g(\underline{x}) - g(\underline{0}) = \int_0^1 \frac{d}{dt} g(t\underline{x})\, dt = \int_0^1 \sum x_i \frac{\partial g}{\partial x_i}(t\underline{x})\, dt$$

---

* for $\mathcal{U}$ not open, it's the set of fcns. with $C^s$ extension to some open $\supset \mathcal{U}$

$\Rightarrow \quad g(\underline{x}) = g(\underline{0}) + \sum x_i G_i(\underline{x}) \quad , \quad G_i(\underline{0}) = \frac{\partial g}{\partial x_i}(\underline{0})$

$\Rightarrow \tilde{D}(g) = \sum \tilde{D}(x_i)\, G_i(0) + \sum \tilde{D}(G_i) \cancel{x_i(0)}^{\,0} = \sum \tilde{D}(x_i) \frac{\partial g}{\partial x_i}(\underline{0})$

$\Rightarrow \tilde{D}$ is a directional derivative at $\underline{0}$. So

$$\mathrm{Der}_p(M) \cong \mathbb{R}\langle \partial/\partial x_1, \ldots, \partial/\partial x_m \rangle \quad \text{is an } m\text{-dim'l vector space}$$

and we set $\quad T_{M,p} := \mathrm{Der}_p(M) \;, \quad T_M^{(\vee)} := \bigcup_{p \in M} T_{M,p}^{(\vee)}$ (co)tangent bundles

To "topologize" these, we turn to the

Definition 2: (i) A real vector bundle (of rank $r$ & class $C^{s \leq k}$) over $M$ is a (separable, Hausdorff) topological space $E$ together with a map $\pi: E \to M$, such that

- the fibers $E_x = \pi^{-1}(x)$ are real vector spaces of dim. $r$
- $\exists$ op. cover $\{U_\alpha\}$ and class $C^s$ homeos. $\Phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^r$ s.t.
  - (a) $pr_{U_\alpha} \circ \Phi_\alpha = \pi|_{U_\alpha}$
  - (b) $\forall x \in U_\alpha$, $pr_{\mathbb{R}^r} \circ \Phi_\alpha|_{E_x} : E_x \to \mathbb{R}^r$ is an $\cong$ of $\mathbb{R}$-vector spaces

(ii) A (class $C^s$) morphism $\Psi: E \to F$ of v.b.'s $/M$ is a (class $C^s$) morphism of the underlying manifolds s.t.

- $\Psi$ is linear on fibers
- $\pi_F \circ \Psi = \pi_E$

A v.b. is trivial if $\cong$ to $M \times \mathbb{R}^r$.

From the defn 2(i), it follows that the $\Phi_\alpha \circ \Phi_\beta^{-1} : \Phi_\beta(\pi^{-1}(U_{\alpha\beta})) \to \Phi_\alpha(\pi^{-1}(U_{\alpha\beta}))$ are actually $C^s$ functions

i.e. matrix entries are

$$\boxed{(B.1)} \begin{cases} \Phi_{\alpha\beta} : U_{\alpha\beta} \to GL(r, \mathbb{R}) \\ \text{s.t. } \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} = \Phi_{\alpha\gamma} \quad \text{on } U_{\alpha\beta\gamma} \end{cases}$$ ("cocycle condition")

Moreover, $E$ is a $C^s$-manifold; and we can perform all the operations of linear algebra on the $\Phi_{\alpha\beta}$ hence on $E$ — this yields bundles $\Lambda^k E$, $E^\vee$, etc.

Proposition 1: $T_M$ is a $C^{k-1}$ bundle over $M$.

Proof: The essential point here is that
$$\Phi_{\alpha\beta} = (\phi_{\alpha\beta})_* \left( := \frac{\partial(x_1^\alpha, \dots, x_m^\alpha)}{\partial(x_1^\beta, \dots, x_m^\beta)} \text{ in basis} \right). \quad \square$$

Definition 3: A (class $C^s$) section of a vector bundle, written $\sigma \in C^s(M, E)$, is a ($C^s$) morphism $\sigma : M \to E$ s.t. $\pi \circ \sigma = id_M$.

Example 1: sections of $\begin{cases} T_M \\ \Lambda^k T_M^\vee \end{cases}$ are called $\begin{cases} \text{vector fields} \\ \text{differential } k\text{-forms} \end{cases}$.

Differential of a fcn.: consider $\xi = \sum \xi_j \partial/\partial x_j \in T_{M,p}$
$f \in C^s(p, \mathbb{R})$

Then $df_p(\xi) := \xi(f) = \sum \xi_j \partial f/\partial x_j (p)$ defines $df_p \in T_{M,p}^\vee$.

In particular, $dx_j(\xi) = \xi_j \Rightarrow \{dx_j\} = \{\partial/\partial x_j\}^\vee$ in each $T_{M,p}^\vee$,

(B.2) and $\boxed{df = \sum \frac{\partial f}{\partial x_j} dx_j \in C^{s-1}(p, T_M^\vee)}$.

Lie bracket: consider $\xi, \eta \in C^s(\mathcal{U}, T_M)$

$$\underbrace{[\xi, \eta]}_{\in C^{s-1}(\mathcal{U}, T_M)}(f) := \xi(\eta(f)) - \eta(\xi(f))$$

In coordinates, $$[\xi, \eta] = \sum_{1 \le j, k \le m} \left( \xi_j \frac{\partial \eta_k}{\partial x_j} - \eta_j \frac{\partial \xi_k}{\partial x_j} \right) \frac{\partial}{\partial x_k}$$

wedge/contraction: given $\omega \in C^s(M, \Lambda^p T_M^\vee)$ $p$-form
$$\omega = \sum_{\substack{|I| = p \\ \text{multiindex}}} \omega_I \underbrace{dx_I}_{dx_{i_1} \wedge \dots \wedge dx_{i_p}}$$
and $\begin{cases} \xi & \text{vector field} \\ \mu & q\text{-form} \end{cases}$

$\mu \wedge \omega$ is a $(p+q)$-form

$\xi \lrcorner \omega$ is a $(p-1)$-form

Ex/ • $\frac{\partial}{\partial x_j} \lrcorner dx_I = \begin{cases} 0, & j \notin I \\ (-1)^{l-1} dx_{I \setminus \{j\}}, & j = i_l \in I \end{cases}$

• $\xi \lrcorner (\mu \wedge \omega) = (\xi \lrcorner \mu) \wedge \omega + (-1)^q \mu \wedge (\xi \lrcorner \omega)$ //

exterior derivative: $d\omega :=_{\text{loc. coords.}} \sum_{\substack{|I|=p \\ 1\le k\le m}} \frac{\partial \omega_I}{\partial x_k} dx_k \wedge dx_I$ defines

$$d : C^s(M, \Lambda^p T_M^\vee) \to C^{s-1}(M, \Lambda^{p+1} T_M^\vee)$$

Ex/ For $\xi^0, \dots, \xi^p$ vector fields,

$$d\omega(\xi^0, \dots, \xi^p) = \sum_{0\le j\le p} (-1)^j \xi_j \cdot \omega(\xi_0, \dots, \hat{\xi_j}, \dots, \xi_p)$$
$$+ \sum_{0\le j<k\le p} (-1)^{j+k} \omega([\xi_j, \xi_k], \xi_0, \dots, \hat{\xi_j}, \dots, \hat{\xi_k}, \dots, \xi_p)$$

"omit"

shows $d\omega$ is intrinsically defined. //

We also have $d(\mu \wedge \omega) = d\mu \wedge \omega + (-1)^q \mu \wedge d\omega$

and $(s \ge 2)$ $d^2 = 0$ $\Leftarrow$ equality of 2nd partials: $\left\{ \begin{array}{l} \text{e.g. } d(df) = \\ d(\sum_i \partial f/\partial x_i \, dx_i) = \\ \sum_i \sum_j \partial^2 f/\partial x_j \partial x_i \, dx_j \wedge dx_i = 0 \\ \text{since } dx_i \wedge dx_j = -dx_j \wedge dx_i . \end{array} \right\}$

pullback: given $F \in C^\infty(M,N)$, $\nu =_{\text{loc}} \sum \nu_J(y)\, dy_J \in C^\infty(N, \Lambda^p T_N^\vee)$ ($dy_J = dy_{j_1} \wedge \dots \wedge dy_{j_p}$)

define the pullback $F^*\nu :=_{\text{loc}} \sum \nu_J(F(\underline{x}))\, dF_{j_1}(\underline{x}) \wedge \dots \wedge dF_{j_p}(\underline{x})$.

The chain rule is (for $G \in C^\infty(M', M)$) $G^* F^* \nu = (F\circ G)^* \nu$, and shows this is invariant w.r.t. choice of coordinates. We also have $\boxed{d \circ F^* = F^* \circ d}$ (†)

de Rham cohomology: set $K^p := C^\infty(M, \Lambda^p T_M^\vee)$. Since $d^2 = 0$,

$(K^\bullet, d) := \dots \to K^{p-1} \xrightarrow{d^{(p-1)}} K^p \xrightarrow{d^{(p)}} K^{p+1} \to \dots$ is a complex,

$(\operatorname{im} d^{(p-1)} \subset \ker d^{(p)})$.

We define (B.3)

$$\boxed{H^p_{dR}(M, \mathbb{R}) := \frac{\ker(d^{(p)})}{\operatorname{im}(d^{(p-1)})} = \frac{\text{closed forms}}{\text{exact forms}}}$$

(degree $p$). By (†), we have $F^* : H^p_{dR}(N, \mathbb{R}) \to H^p_{dR}(M, \mathbb{R})$.

($H^p_{dR,c}(M, \mathbb{R})$ is obtained by doing everything w/ compact support, for $M$ noncompact this gives a different result.)

So there's your differential calculus.

Now we briefly review a bit of integral calculus.

**Definition 4**: Let $\omega \in C^0_c(\mathcal{U}, \Lambda^m T^\vee_M)$, $\mathcal{U} \subset U_\alpha$.

Then $\int_M \omega := \int_{\mathbb{R}^m} f(\underline{x})\, dx_1^\alpha \wedge \cdots \wedge dx_m^\alpha$ is independent of coordinates by the $\Delta$ of variables formula.

loc

If $\mathcal{U}$ is replaced by $M$, and $\{g_\alpha\}$ is a partition of unity ($\sum g_\alpha \equiv 1$ on $M$)

$$\int_M \omega = \sum \int_{U_\alpha} g_\alpha \omega \quad \text{is well-defined.}$$

$F \in C^1(N, M) \to \|$

$$(\pm)\int_N F^* \omega$$ with sign acc. to whether $\det(\text{Jacobian of } F) > 0$.
(i.e. orientation preserved)

**Stokes theorem**: Let $\omega \in C^1(M, \Lambda^{m-1} T^\vee_M)$, and

$\Gamma \subset M$ be a compact subset with ~~piecewise~~ $C^1$ boundary.

Then $\int_{\partial\Gamma} \omega = \int_\Gamma d\omega$.

wolog some locally of form $x_1, \dots, x_\ell \le 0$ in coords on $U$

**Sketch of pf.**: locally $\omega = \sum f_j \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_m$

$\partial_j \Gamma \cap U = \partial\Gamma \cap \{x_j = 0\} \cap U$

If $\omega$ comp. supp. on $U$,

$$\int_{\partial_j \Gamma} \omega = \int_{\Gamma \cap U} \frac{\partial f_j}{\partial x_j} dx_1 \wedge \cdots \wedge dx_m \qquad (\text{partially } \textstyle\int \text{ wrt } x_j)$$

$$\int_{\partial\Gamma} \omega = \sum (-1)^{j-1} \int_{\partial_j \Gamma} \omega = \int_{\Gamma \cap U} \underbrace{\sum (-1)^{j-1} \frac{\partial f_j}{\partial x_j} dx_1 \wedge \cdots \wedge dx_m}_{d\omega}$$

General result follows from a partition of unity argument. $\square$

Now let $\Pi : [0,1] \times M \to M$, $(t, p) \mapsto p$, $\quad I_t : M \hookrightarrow [0,1] \times M$, $p \mapsto (t, p)$

and define

$$\Pi_* : C^s\left([0,1]\times M, \Lambda^r T^\vee_{[0,1]\times M}\right) \to C^s\left(M, \Lambda^{r-1} T^\vee_M\right)$$

by integrating along the fiber. (This is called "push-forward", and kills all terms of type $dx_I$ — only $dt \wedge dx_I$'s go somewhere.)

Ex/ $\Pi_* d\omega + d\Pi_* \omega = I_1^* \omega - I_0^* \omega$ (essentially FTC with cancellations)

Observation: If $F, G \in C^\infty(M,N)$ are smoothly homotopic

$$\left(\exists\, H \in C^\infty([0,1]\times M, N) \text{ s.t. } H\circ I_0 = F,\ H\circ I_1 = G\right)$$

then

$$G^*\omega - F^*\omega = (I_0^* - I_1^*)(H^*\omega)$$
$$= \Pi_* \underbrace{dH^*\omega}_{\substack{H^* d\omega \\ \shortparallel \\ 0 \text{ if } \omega \text{ closed}}} + \underbrace{d(\Pi_* H^*\omega)}_{\text{exact}}.$$

$$\Rightarrow \boxed{F^* = G^* : H^p_{dR}(N,\mathbb{R}) \to H^p_{dR}(M,\mathbb{R})}$$

Example 2: $N = M$ contractible, $F(M) = \{x_0\}$, $G = Id_M$

$$\Rightarrow H^p_{dR}(M,\mathbb{R}) = \begin{cases} \mathbb{R}, & p=0 \\ 0, & p>0 \end{cases}.$$

As a corollary, we obtain the

d-Poincaré Lemma: Given $\mathcal{U} \subset \mathbb{R}^n$ starshaped open set ($\Rightarrow$ contractible), $\omega \in C^s(\mathcal{U}, \Lambda^p T^\vee_{\mathcal{U}})$ $(p \geq 1)$ closed $\Rightarrow$ $\omega$ exact. $(= d\eta)$

We conclude with some further examples of de Rham cohomology groups.

Example 3: $\dim H^0_{dR}(M,\mathbb{R}) = \#\{\text{connected components of } M\}$.

[Proof: $H^0_{dR} = \ker(d^{(0)}) =$ constant functions.]

Example 4: $H^2_{dR}(S^2,\mathbb{R}) \cong \mathbb{R}$

$S^2$

Proof: Let $\iota: S^2 \hookrightarrow \mathbb{R}^3$ (~~w~~ coords $x_1, x_2, x_3$)

$\theta := \sum_j x_j \, \partial/\partial x_j$

$\Omega := dx_1 \wedge dx_2 \wedge dx_3$ ;

then $\xi := \iota^*(\theta \lrcorner \Omega)$ is $\begin{cases} \text{nonzero} \\ \text{closed (since of top degree)} \\ SO(3,\mathbb{R})\text{-invariant} \end{cases}$,

and is the unique such form (up to scale).

NOTE: Averaging does not change coh. class, because any $g^*$ is homotopic to identity; moreover, averaging kills any form with zero $\int_{S^2}$.

If $\xi = d\omega$, for $\omega \in C^\infty(S^2, T^\vee_{S^2})$, then for $g \in SO(3,\mathbb{R})$

$(\xi =)\, g^*\xi = g^*(d\omega) = d(g^*\omega)$

$\Rightarrow \xi = d\Big(\underbrace{\int_{SO(3,\mathbb{R})} g^*\omega \, d\mu}\Big)$ Haar measure: $\int_{SO(3)} d\mu = 1$, $g^* d\mu = d\mu$ $(\forall g)$ (for compact Lie gp)

is $SO(3)$-invariant, in particular invariant under reflection through any line through the origin (which is $-\mathrm{id}$ on $T^\vee_{S^2,\, \ell \cap S^2}$)

$= d(0) = 0$, a contradiction. $\square$

Ex/Example 5: using a group action one can also show

(a) $H^1_{dR}(S^2,\mathbb{R}) = \{0\}$

(b) $\dim H^*_{dR}(\mathbb{R}^2/\mathbb{Z}^2, \mathbb{R}) = \begin{cases} 1, & * = 0 \\ 2, & * = 1 \\ 1, & * = 2 \end{cases}$

dim(M)

Example 6: If $M$ is compact & connected, then $\dim H^m_{dR}(M,\mathbb{R}) = 0$ or $1$,

and $M$ orientable $\iff \dim H^m_{dR}(M,\mathbb{R}) = 1$.

Ex/ Use a partition of unity + Stokes theorem to prove:

$M$ orientable $\Longrightarrow H^m_{dR}(M,\mathbb{R}) \neq \{0\}$.