C. Complex manifolds

We begin with a modest dose of "several complex variables".

Let $\mathcal{D} \subseteq \mathbb{C}^n$ be a connected open set, coords. $z_j = x_j + i y_j$,
$f : \mathcal{D} \to \mathbb{C}$, $f = u + iv$. (no assumptions on the function f)

Definition 1: (a) f analytic $\iff$ $\forall\, \underline{\alpha} \in \mathcal{D}$ $\exists$ open $\mathcal{U} \subset \mathcal{D}$
s.t. $(f|_{\mathcal{U}})(\underline{z}) = \underbrace{\sum_{\underline{k} \in \mathbb{Z}^n_{\geq 0}} c_{\underline{k}} \cdot (\underline{z} - \underline{\alpha})^{\underline{k}}}_{\text{(convergent in } \mathcal{U})}$

(b) f holomorphic $\iff$ f separately holomorphic in each z_j
written "$f \in \mathcal{O}(\mathcal{D})$" (with other $\{z_{j'}\}$ held fixed)

Hartogs theorem: $(a) \underset{\text{easy}}{\iff} (b)$

$(b) \to (a)$
Sketch: Assuming f locally bounded (otherwise it's quite hard), iteratively applying 1-variable Cauchy gives

$$\text{(C.1)} \quad \boxed{\text{"polydisk Cauchy"} \quad f(\underline{z}) = \frac{1}{(2\pi i)^n} \int_{\hat{\times}_{j=1}^{n} \{|\zeta_j - z_j| = r_j\}} \frac{f(\underline{\zeta})\, d\zeta_1 \wedge \cdots \wedge d\zeta_n}{\prod_j (\zeta_j - z_j)}}$$

(The local boundedness comes in at the end, in rearranging the $\int \cdots \int$.)
Expanding the $\frac{1}{\zeta_j - z_j} = \frac{1/\zeta_j}{1 - z_j/\zeta_j}$ in power series completes the proof. $\square$

More generally, if f is just C^1, then we write

$$df = \sum \frac{\partial f}{\partial x_j} dx_j + \sum \frac{\partial f}{\partial y_j} dy_j = \underbrace{\sum \overbrace{\tfrac{1}{2}\left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j}\right)}^{\partial/\partial z_j} f \cdot \overbrace{(dx_j + i\, dy_j)}^{dz_j}}_{\partial f} + \underbrace{\sum \overbrace{\tfrac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}\right)}^{\partial/\partial \bar{z}_j} f \cdot \overbrace{(dx_j - i\, dy_j)}^{d\bar{z}_j}}_{\bar{\partial} f}$$

which suggests "$d = \partial + \bar{\partial}$" (more on this in a moment).

We may view $df|_{p\in\mathcal{D}}$ as the map $f_*|_p : T_{\mathcal{U},p} \to T_{\mathbb{R}^2, f(p)}$) viewing these both as real diff. manifolds

given w.r.t. bases $\{\{\partial/\partial x_j\}, \{\partial/\partial y_j\}\}$ and $\{\partial/\partial u, \partial/\partial v\}$ by

$$\begin{pmatrix} u_{x_1} & u_{y_1} & \cdots & u_{x_n} & u_{y_n} \\ v_{x_1} & v_{y_1} & \cdots & v_{x_n} & v_{y_n} \end{pmatrix}.$$

This element of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^2)$ is $\underline{\mathbb{C}\text{-linear}}$, i.e. comes from $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$,

iff the matrix is of the form $\begin{pmatrix} \alpha_1 & -\beta_1 & \cdots & \alpha_n & -\beta_n \\ \beta_1 & \alpha_1 & & \beta_n & \alpha_n \end{pmatrix}$ (i.e., CR equations hold).

We conclude that

$$\text{(C.2)} \quad \boxed{f \in \mathcal{O}(\mathcal{D}) \iff \bar{\partial} f = 0 \iff f_*|_p \ \mathbb{C}\text{-linear } (\forall p \in \mathcal{D})}$$

Besides (C.1–2) there are a lot of

<u>similarities w./1-variable case</u>:

(i) $\underset{\text{sequence}}{\{f_k\}} \subset \mathcal{O}(\mathcal{D})$ locally uniformly bounded $\implies$

$\exists$ normally convergent subsequence $\{f_{k_\ell}\}$, and then $\lim_{\ell\to\infty} f_{k_\ell} \in \mathcal{O}(\mathcal{D})$.

(ii) $\left.\begin{array}{l} \mathcal{U} \subset \mathcal{D} \text{ open/nonempty} \\ f \in \mathcal{O}(\mathcal{D}),\ f|_{\mathcal{U}} \equiv 0 \end{array}\right\} \implies f \equiv 0$ [extend in each variable along slices]

(iii) $\left.\begin{array}{l} f \in \mathcal{O}(\mathcal{D}),\ \underline{\alpha} \in \mathcal{D} \\ |f(\underline{\alpha})| \geq |f(\underline{z})| \ \ \forall \underline{z} \in \mathcal{D} \end{array}\right\} \implies f$ constant [use Cauchy – see Voisin]

(iv) $\left.\begin{array}{l} f \in \mathcal{O}(\mathcal{D} \setminus \{z_1 = 0\}) \\ f \text{ bounded in nbhd. of } \{z_1 = 0\} \end{array}\right\} \implies f$ extends to $\mathcal{O}(\mathcal{D})$

<u>Pf for n=2</u>: near $(0,0) \in \mathcal{D}$ (say), $f(z_1, z_2) = \sum_{j=-\infty}^{\infty} \alpha_j(z_2) z_1^j$

where $\alpha_j(z_2) = \frac{1}{2\pi i} \oint_{|\zeta|=\epsilon} \frac{f(\zeta, z_2)}{\zeta^{j+1}} d\zeta$ is holomorphic by Morera (1-var.).

We then get $f = \sum_{\substack{j \in \mathbb{Z} \\ k \geq 0}} \alpha_{jk} z_1^j z_2^k$ and it's then easy to see

(using boundedness now) that $\alpha_{jk} = 0$ for $j < 0$. $\square$

Ex/ Liouville on $\mathbb{C}^n$.

What's different:

(v) no analogue of Riemann mapping thm. — $(\text{unit disk})^{\times n}$ & (unit n-ball) are not biholo. [Poincaré]

(vi) largest domain of convergence of a power series not a polydisk (or ball) in general

(vii) Hartogs phenomenon: ($n \geq 2$ only)

$$\left.\begin{array}{l} K \subset D \text{ compact} \\ D \setminus K \text{ connected} \\ f \in \mathcal{O}(D\setminus K) \end{array}\right\} \Rightarrow f \text{ extends to } \mathcal{O}(D).$$

Pf. for $n=2$, $K=\{(0,0)\}$: follow 1st 2 lines of pf. of (iv) above. Then have $\alpha_j(z_2) = 0$ for $j<0$ & $z_2 \neq 0$, since there f is a holo. function of z_1 on a disk about 0. By (ii), $\alpha_{j<0} \equiv 0$. $\square$

1-variable $\bar\partial$-Poincaré Lemma: let $K \subset \mathbb{C}$ be compact w./piecewise C^1 boundary, $f \in C^1(K, \mathbb{C})$ (i.e. $U \supset K$). Then by Stokes theorem, for $z \in \text{int}(K)$

$$\int_K \frac{\partial f/\partial \bar w (w)}{w-z}\, d\bar w \wedge dw = \lim_{\epsilon\to 0} \int_{K\setminus D_\epsilon(z)} d\left(\frac{f(w)}{w-z}\, dw\right) = \underbrace{-\lim_{\epsilon\to 0}\int_{\partial D_\epsilon(z)} \frac{f(w)}{w-z}\, dw}_{2\pi i f(z)} + \int_{\partial K} \frac{f(w)}{w-z}\, dw$$

which gives a generalization of Cauchy. If f is compactly supported in K, then the $\int_{\partial K} = 0$, and this becomes

essential for these $\int$'s to converge

$$f(z) = \frac{1}{2\pi i}\int_K \frac{\partial f/\partial \bar w (w)}{w-z}\, dw \wedge d\bar w \underset{\Delta \text{ of var.}}{=} \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial f/\partial \bar w (w+z)}{w}\, dw \wedge d\bar w$$

$$= \frac{1}{2\pi i}\frac{\partial}{\partial \bar z}\int_{\mathbb{C}} \frac{f(w+z)}{w}\, dw \wedge d\bar w \overset{\downarrow}{=} \frac{\partial}{\partial \bar z}\underbrace{\left(\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(w)}{w-z}\, dw \wedge d\bar w\right)}_{=:u}.$$

NOTE: they have integrable singularities

We conclude the

Theorem 1: Given a 1-form $f(z)\,d\bar z$ (of class C^s) compactly supported on $\mathbb{C}$, $\exists\ u \in C^s(\mathbb{C}, \mathbb{C})$ satisfying $\boxed{\bar\partial u = f(z)\, d\bar z}$.

Finally, we have the

<u>Holomorphic Inverse Function Theorem</u>: Let $U, \overset{\overset{p}{\cap}}{V} \underset{op}{\subset} \mathbb{C}^n$ and $F \in \mathcal{Hol}(U,V)$, i.e. the $F_j \in \mathcal{O}(U)$. <u>If $F_*|_p$ is nonsingular</u> ($\det F_*|_p \neq 0$) then $\exists\ \underset{p\in}{U'} \subset U,\ \underset{F(p)\in}{V'} \subset V$ s.t. $F|_{U'} : U' \to V'$ is biholomorphic. //

<u>Remark</u>: More generally, if $V \subset \mathbb{C}^m$ and $F_*|_p$ has rank k everywhere, the <u>rank theorem</u> says that (up to a local biholomorphic change of coordinates) F takes the form $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_k, 0, \dots, 0)$ locally.

Now we can get to business: the holo. analogue of Defns. B.1–2.

<u>Definition 2</u>: (a) A <u>complex manifold</u> M of (complex) dimension n is a differentiable manifold (of real dim. $2n$) having <u>biholomorphic transition functions</u>* $\phi_{\alpha\beta}$. In particular, all $(\phi_{\alpha\beta})_*|_p$ are <u>$\mathbb{C}$-linear</u>.

(b) $\mathcal{O}(M)$ and $\mathcal{Hol}(M,N)$ are also easy tweaks to the previous definitions.

(c) A <u>holomorphic vector bundle</u> $E \xrightarrow{\pi} M$ of (complex) rank r, is defined as in Defn. B.2, but

- taking $\Phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\simeq} U_\alpha \times \mathbb{C}^r$ to be biholomorphic
- taking $pr_{\mathbb{C}^r} \circ \Phi_\alpha|_{E_p} : E_p \to \mathbb{C}^r$ to be $\cong$'s of $\mathbb{C}$-vector spaces.

Equivalently, the transition functions $\Phi_{\alpha\beta}$ map $U_{\alpha\beta} \to GL(r, \mathbb{C})$ and are holomorphic. $\square$

$\cap$ proper subgp. $GL(2r, \mathbb{R})$

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* obviously, we have to identify the $\mathbb{R}^{2n} \supset V_\alpha$ with $\mathbb{C}^n$ for this to make sense

Remark 2: Any such M is orientable, since

$$\det(\phi_{\alpha\beta})_* = \det\begin{pmatrix} A & -B \\ B & A \end{pmatrix} = |\det(A+Bi)|^2 > 0. \quad (\text{cf. } (A.5)).$$

Ex/ Show that for M compact, complex, connected & $f \in \mathcal{O}(M)$, f is constant. //

Given a complex manifold M, the "underlying" real manifold has a tangent bundle (of rank $2n$). This has the structure of a holomorphic vector bundle T_M, since $\Phi_{\alpha\beta} = (\phi_{\alpha\beta})_*$ are complex-linear (ptwise) and have holomorphic matrix coefficients. By (A.4) we can think of this equivalently as the underlying real bundle $T_M^{\mathbb{R}}$ together with $J \in C^\infty(M, \mathrm{End}(T_M^{\mathbb{R}}))$ satisfying $J^2 = -id$ (corresponding to "mult. by $i = \sqrt{-1}$" on each $T_{M,p}$). The chart

i.e. $(T_M^{\mathbb{R}})^\vee \otimes T_M^{\mathbb{R}}$

ϕ_α gives (local) coordinates $(z_1^{(\alpha)}, \ldots, z_n^{(\alpha)})$ on $\mathcal{U}_\alpha \subset M$. As before we have the identification

$$(T_{M,p} =)\ T_{M,p}^{\mathbb{R}} = \mathbb{R}\langle \partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/\partial x_n, \partial/\partial y_n \rangle.$$

Though this has a complex structure, we don't have yet $\partial/\partial z_j$ & $\partial/\partial \bar{z}_j$. To get this we must consider the "complexified tangent bundle" $T_M \otimes_{\mathbb{R}} \mathbb{C}$ of complex rank $2n$:

(C.3)
$$\begin{aligned} T_{M,p}^{\mathbb{R}} \otimes \mathbb{C} &= \mathbb{C}\langle \partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/\partial x_n, \partial/\partial y_n \rangle \\ &= \mathbb{C}\langle \underbrace{\partial/\partial z_1, \ldots, \partial/\partial z_n}\, ; \underbrace{\partial/\partial \bar{z}_1, \ldots, \partial/\partial \bar{z}_n} \rangle \\ &=: T_{M,p}^{1,0} \oplus T_{M,p}^{0,1} \end{aligned}$$

<u>Proposition 1</u>: $T_M^{1,0}$ is a (rank n) holomorphic vector bundle.

<u>Proof</u>: $v \mapsto v - iJ(v)$ induces an isomorphism $T_M \xrightarrow{\cong} T_M^{1,0}$.

$$\Big(\text{It is } \mathbb{C}\text{-linear via } \underset{\substack{\text{mult. by } i \\ \text{on } T_M}}{J}(v) \mapsto J(v) - i\underbrace{J(J(v))}_{-v} = J(v) + iv = i(v - iJ(v)). \Big) \qquad \square$$

By duality we have a decomposition

$$T_M^\vee \otimes_{\mathbb{R}} \mathbb{C} \cong (T_M \otimes_{\mathbb{R}} \mathbb{C})^\vee \cong \underset{\substack{\uparrow \\ \text{loc. gen.} \\ dz_1, \dots, dz_n}}{T_M^{\vee(1,0)}} \oplus \underset{\substack{\uparrow \\ \text{loc. gen} \\ d\bar{z}_1, \dots, d\bar{z}_n}}{T_M^{\vee(0,1)}} \tag{C.4}$$

In coordinates (at some p):

$$dz_j(\partial/\partial z_k) = \delta_{jk}, \quad dz_j(\partial/\partial \bar{z}_k) = 0$$

$$d\bar{z}_j(\partial/\partial z_k) = 0, \quad d\bar{z}_j(\partial/\partial \bar{z}_k) = \delta_{jk}.$$

Another reason why you'd want to introduce $T_M^\vee \otimes_{\mathbb{R}} \mathbb{C}$: we have

$$d : C^\infty(M, \mathbb{R}) \to C^\infty(M, T_M^\vee).$$

To differentiate cx.-valued fns., we have no choice! That is,

$$\boxed{d : C^\infty(M, \mathbb{C}) \longrightarrow C^\infty(M, T_M^\vee \otimes_{\mathbb{R}} \mathbb{C})} \tag{C.5}$$

In coordinates this allows us to write (computing as on p. 10)

$$\boxed{df = \sum \frac{\partial f}{\partial z_j^\alpha} dz_j^\alpha + \sum \frac{\partial f}{\partial \bar{z}_j^\alpha} d\bar{z}_j^\alpha} \tag{C.6}$$

[not valid in $T_M^\vee$ — only $T_M^\vee \otimes_{\mathbb{R}} \mathbb{C}$].

More generally:

<u>Definition 3</u>: The elements of

$$A^k(M) := C^\infty(M, \Lambda^k(T_M^\vee \otimes_{\mathbb{R}} \mathbb{C}))$$

are called <u>complex-valued differential k-forms</u> on M. //

Now by (A.1),

(C.7)
$$\Lambda^k(T_M^\vee \otimes_{\mathbb{R}} \mathbb{C}) = \Lambda^k\left(T_M^{\vee(1,0)} \oplus T_M^{\vee(0,1)}\right) = \bigoplus_{p+q=k} \underbrace{\Lambda^p T_M^{\vee(1,0)} \otimes \Lambda^q T_M^{\vee(0,1)}}_{\Lambda^{p,q} T_M^\vee}$$

and accordingly

(C.8)
$$\boxed{A^k(M) \cong \bigoplus_{p+q=k} C^\infty(M, \Lambda^{p,q}T_M^\vee) =: \bigoplus_{p+q=k} \underbrace{A^{p,q}(M)}_{=:\ \underline{\text{forms of } (p,q)\text{-type}}}.}$$

A given $\omega \in A^{p,q}(M)$ may be written locally (in $U \subset M$ open) as

(C.9)
$$\| \quad \sum_{|I|=p,\,|J|=q} \omega_{I,J}(z)\, dz_I \wedge d\bar{z}_J \quad , \qquad \omega_{I,J} \in C^\infty(U,\mathbb{C}).$$

Now (C.6) shows that on functions, the exterior derivative splits into (1,0) & (0,1) parts; viz., for $f \in C^\infty(M,\mathbb{C})$,

$$df = \partial f + \bar\partial f,$$

and

(C.10)
$$\boxed{f \in \mathcal{O}(M) \iff \bar\partial f = 0 \iff df \in C^\infty(M, T_M^{\vee(1,0)})}$$

in which case df is actually a <u>holomorphic</u> section of $T_M^{\vee(1,0)}$. This splitting $\boxed{d = \partial + \bar\partial}$ can be extended to $\mathbb{C}$-valued forms:

$$\partial\omega := \sum_{I,J} \sum_{\ell=1}^{n} \frac{\partial \omega_{I,J}}{\partial z_\ell}\, dz_\ell \wedge dz_I \wedge d\bar{z}_J \quad ,$$

ω in (C.9), say

$$\bar\partial\omega := \sum_{I,J} \sum_{\ell=1}^{n} \frac{\partial \omega_{I,J}}{\partial \bar{z}_\ell}\, d\bar{z}_\ell \wedge dz_I \wedge d\bar{z}_J \quad .$$

giving maps

$$(C.11)\qquad \boxed{\begin{aligned} \partial &: A^{p,q}(M) \longrightarrow A^{p+1,q}(M) \\ \bar\partial &: A^{p,q}(M) \longrightarrow A^{p,q+1}(M) \end{aligned}},$$

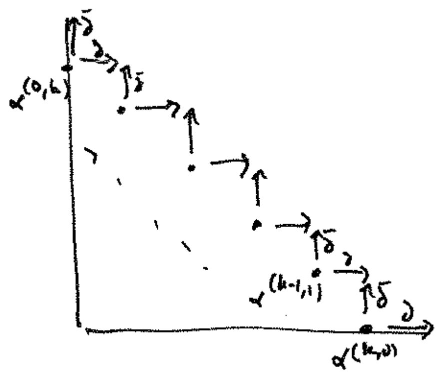
and then

$$0 = d^2 = (\partial + \bar\partial)^2 = \underbrace{\partial^2}_{A^{p+2,q}} + \underbrace{(\partial\bar\partial + \bar\partial\partial)}_{A^{p+1,q+1}} + \underbrace{\bar\partial^2}_{A^{p,q+2}}$$

so that (in our first argument "by type")

$$(C.12)\qquad \boxed{\partial^2 = (\partial\bar\partial + \bar\partial\partial) = \bar\partial^2 = 0.}$$

<u>Remark 3:</u> It's useful to have the following picture of what it means for a k-form $\alpha = \sum_{p+q=k} \alpha^{(p,q)}$ to be d-closed:



we must have $\partial\alpha^{(k,0)} = 0 = \bar\partial\alpha^{(0,k)}$ and $\partial\alpha^{(p,q)} = -\bar\partial\alpha^{(p+1,q+1)}$ for each (p,q).

$\square$

We also have the following analogue of de Rham cohomology: setting

$$\begin{aligned} &K^{q-1} := A^{p,q-1}(M) \\ &\downarrow \bar\partial^{(q-1)} \\ &K^{q} := A^{p,q}(M) \\ &\downarrow \bar\partial^{(q)} \\ &K^{q+1} := A^{p,q+1}(M) \end{aligned}$$

we have

Definition 4: Dolbeault cohomology is $\boxed{H^{p,q}_{\bar\partial}(M,\mathbb{C}) := \dfrac{\ker \bar\partial^{(q)}}{\operatorname{im} \bar\partial^{(q-1)}}}$.

Writing (C.13) $\boxed{\Omega^p(M) := \mathcal{O}(M, \Lambda^p T_M^{\vee(1,0)})}$ holo. bundle (so holo. sections make sense)

for holomorphic forms — these are locally of the form $\sum \omega_I(z) dz_I$, $\omega_I \in \mathcal{O}(U)$ — we prove the

Proposition 2: $\Omega^p(M) \cong H^{p,0}_{\bar\partial}(M,\mathbb{C}) \quad (\forall p)$

Proof: Given $\omega \in \Omega^p(M)$, $\bar\partial\omega = 0$.

Conversely, let

$$\alpha \in A^{p,0}(M) \quad \text{with} \quad \bar\partial\alpha = 0$$

$\alpha \overset{\text{loc (in } U)}{=} \sum_I \alpha_I(z)\, dz_I$, and

$$\bar\partial\alpha \overset{\text{loc}}{=} \sum_I \sum_{j=1}^n \frac{\partial \alpha_I}{\partial \bar z_j} \underbrace{d\bar z_j \wedge dz_I}_{\neq 0}$$

$$\Rightarrow \text{all } \frac{\partial \alpha_I}{\partial \bar z_j} = 0$$

$$\Rightarrow \text{all } \alpha_I \in \mathcal{O}(U)$$

So $\alpha \in \Omega^p(M)$. Finally, note that $\alpha = \bar\partial\beta$ is impossible "by type". □

Remark 4: It is also true that $\Omega^p(M) \subset H^p_{dR}(M,\mathbb{C})$. But without the Hodge theorem all we can show is that $\omega \in \Omega^n(M)$ defines a class in $H^n_{dR}(M,\mathbb{C})$ (not even that the resulting map $\Omega^n(M) \to H^n_{dR}(M,\mathbb{C})$ is injective). This is done by observing

$\bar\partial\omega = 0$ and (since wedging $(n+1)$ dz's together gives 0) $\partial\omega = 0$

$\Rightarrow d\omega = 0.$ □

Remark 5: Given $F \in \underline{\underline{Hol}}(M,N)$ (cx-mflds.), $\boxed{F^* : H^{p,q}_{\bar\partial}(N,\mathbb{C}) \to H^{p,q}_{\bar\partial}(M,\mathbb{C})}$

is defined. This is because $\begin{cases} F^* dw_k = dF_k & \text{is in } T^{\vee(1,0)} \\ F^* d\bar w_k = d\bar F_k & \text{is in } T^{\vee(0,1)} \end{cases}$ (holo.)

$\Rightarrow F^*$ preserves type (FALSE for C^∞ F). So, for a form α

(C^∞ of type (p,q))

$$\underset{(p+1,q)}{\partial F^*\alpha} + \underset{(p,q+1)}{\bar\partial F^*\alpha} = dF^*\alpha = F^* d\alpha = \underset{(p+1,q)}{F^*\partial\alpha} + \underset{(p,q+1)}{F^*\bar\partial\alpha}$$

$\Rightarrow F^*$ commutes with $\partial, \bar\partial$. □

Ex/ Show that the cup products on Dolbeault cohomology (defined by wedging forms together) are well-defined. //

We now come to the key result of this section:

Theorem 2 ($\bar\partial$-Poincaré Lemma): Let $U \subset \mathbb{C}^n$ be a polydisk,

$q \geq 1$, and $\alpha \in A^{p,q}(U)$ with $\bar\partial\alpha = 0$. Then $\exists$

$\beta \in A^{p,q-1}(U)$ s.t. $\bar\partial\beta = \alpha$. (In other words, $H^{p,q}_{\bar\partial}(U) = 0$ for $q \geq 1$.)

Proof: It will be convenient to choose a nested covering of U:

$U = \bigcup_{k=1}^{\infty} U_k$, $U_k \subset U_{k+1}$; and smooth fcns. on U $\eta_k = \begin{cases} 1 \text{ on } U_k \\ 0 \text{ off } U_{k+1} \end{cases}$

Step 1: $\alpha = \sum_{I,J} \alpha_{I,J}\, dz_I \wedge d\bar{z}_J$ [Note: $|I| = p$, $|J| = q$ throughout.]

$$= \sum_I \Big(dz_I \wedge \underbrace{\sum_J \alpha_{I,J}\, d\bar{z}_J}_{\alpha_I \ \text{type } (0,q)} \Big)$$

$$0 = \bar\partial\alpha = (-1)^p \sum_I dz_I \wedge \bar\partial\alpha_I \implies \bar\partial\alpha_I = 0.$$

If we can prove $\bar\partial$-Poincaré for $(\overset{(,p)}{0}, q)$, then $\alpha_I = \bar\partial\beta_I$ ← type $(0,q-1)$

and $\bar\partial\left((-1)^p \sum dz_I \wedge \beta_I\right) = \sum dz_I \wedge \alpha_I = \alpha$.

Step 2: $\alpha = f\, d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q$, $\bar\partial\alpha = 0 \implies \dfrac{\partial f}{\partial \bar{z}_{\ell > q}} = 0$

(sm. fcn. from above) in q^{th} place

Set $$g(z) := \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{(\eta_1 f)(z_1, \ldots, w, \ldots, z_n)}{w - z_q}\, dw \wedge d\bar{w}$$

$\underset{\text{proof of Thm. (C.1)}}{\implies}$ $\partial g / \partial \bar{z}_q = f$ & $\partial g / \partial \bar{z}_{\ell > q} = 0$ on U_1 (though g is defined on U)

$\implies \bar\partial\left((-1)^{q-1} g\, d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{q-1}\right) = \alpha$ on U_1,

proving $\bar\partial$-Poincaré LOCALLY (i.e. on a smaller nbhd. than started with) for $(0,q)$ forms of a special type.

Step 3: $\alpha = \sum \alpha_J\, d\bar{z}_J$ type $(0,q)$ $\bar\partial$-closed (pf. of general local case)

Induce on $k = k(\alpha) = \max_J \{k \mid k \in J \text{ and } \alpha_J \neq 0\}$ (base case $k = q$ was Step 2).

Assume done (locally) for $k-1$, write

$$\alpha'' = \sum_{J \ni k} \alpha_J\, d\bar{z}_{J \setminus \{k\}}, \qquad \alpha' = \sum_{J \not\ni k} \alpha_J\, d\bar{z}_J$$

$\implies \alpha = \alpha'' \wedge d\bar{z}_k + \alpha'$, with $k(\alpha''), k(\alpha') \leq k-1$.

$\implies 0 = \bar\partial\alpha = \bar\partial\alpha'' \wedge d\bar{z}_k + \bar\partial\alpha'$ ← no terms with $d\bar{z}_k \wedge d\bar{z}_{\ell > k}$

$\Rightarrow \quad \dfrac{\partial \alpha_J}{\partial \bar z_\ell} = 0$ for $J \ni k$ and $\ell > k$

$\Rightarrow \quad \exists \underset{(\text{loc.})}{g_J}$ s.t. $\dfrac{\partial g_J}{\partial \bar z_\lambda} = 0$ & $\dfrac{\partial g_J}{\partial \bar z_k} = \alpha_J$

$\Rightarrow \quad \bar\partial \underbrace{\left(\sum_{J \ni k} g_J \, d\bar z_J \right)}_{=:\beta} = (-1)^{q-1} \alpha'' \wedge d\bar z_k + \tilde\alpha'$ where $k(\tilde\alpha') \le k-1$

$\Rightarrow \quad \alpha = \bar\partial \beta + \underbrace{(\alpha' - \tilde\alpha')}_{}$

$\bar\partial$-closed, apply inductive hypothesis.

Step 4: We have proved Poincaré $\bar\partial$ locally for $\bar\partial$-closed $(0,q)$ forms α.

For any k, we can get ψ_k $(0,q-1)$ form compactly supported on U_{k+1} & s.t. $\bar\partial \psi_k = \alpha$ on U_k. (Use η_k instead of η_1 in the above proof steps, & multiply the solution it produces by η_k.)

So why not take $\lim_k \psi_k$? They might not converge uniformly on compact sets, which is necessary for $\lim_k$ to commute with $\bar\partial$.

It is necessary to modify the ψ_k's; the argument inducts on q:

(i) Assume the global problem (on U) solved for $q-1$. Then

$\bar\partial(\psi_k - \psi_{k+1}) = 0$ on $U_k \Rightarrow$

$\psi_k - \psi_{k+1} = \bar\partial \tilde\beta$ on U_k (truncate to $\beta := \eta_{k-1}\tilde\beta$, so $\psi_k - \psi_{k+1} = \bar\partial\beta$ on U_{k-1})

$\Rightarrow \psi'_{k+1} := \psi_{k+1} + \bar\partial\beta$ satisfies

$\bar\partial \psi'_{k+1} = \bar\partial\psi_{k+1} = \alpha$ on U_{k+1} & agrees with ψ_k on U_{k-1}

(truncate to $\psi''_{k+1} = \eta_{k+1}\psi'_{k+1}$ which in addition to these properties is compactly supp. on U_{k+2})

Proceed to modify the remaining ψ_ℓ (to agree successively w/ $\psi_{\ell-1}$ on $U_{\ell-2}$), done.

(ii) (base case) Rare, that is, unless $q=1$. Then $\bar\partial(\psi_k - \psi_{k+1}) = 0$ on U_k

$\Rightarrow$ HOLOMORPHIC there.

$\Rightarrow$ $\exists$ polynomial β on U approx. it to within $\frac{1}{2^k}$ on U_{k-1}

$\Rightarrow$ $\psi'_{k+1} = \psi_{k+1} + \beta$ satisfies

$\bar\partial \psi'_{k+1} = \bar\partial \psi_{k+1} = \alpha$ & is within $1/2^k$ of ψ_k on U_{k-1}

$\Rightarrow$ $\psi = \lim \psi_k$ exists, satisfies $\bar\partial\psi = \alpha$. $\square$

Let $E \to M$ be a holomorphic vector bundle,

(C.14) $$\boxed{A^{0,q}(M,E) := C^\infty(M, \Lambda^q T_M^{*(0,1)} \otimes E)}$$

Since $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r$, there is a local basis of holo. sections $\{\sigma_j\}_{j=1}^r$ and an arbitrary $\sigma \in C^\infty(M,E)$ is $\sigma = \sum f_j \sigma_j$.

Set

(C.15) $$\boxed{(\bar\partial_E \sigma)\big|_{U_\alpha} := \sum \bar\partial f_j \otimes \sigma_j \in A^{0,1}(U_\alpha, E)}$$

Over U_β, there is a different basis of holomorphic sections $\{\tilde\sigma_k\}$, and on $U_{\alpha\beta}$ $\sigma_j = \sum g_{jk} \tilde\sigma_k$ where the $g_{jk} \in \mathcal{O}(U_{\alpha\beta})$. Consequently $\bar\partial\sigma$ will not depend (in $U_{\alpha\beta}$) on the choice of trivialization, and so

$$\bar\partial_E \sigma \in A^{0,1}(M,E)$$

is defined. Clearly for a given $\sigma \in C^\infty(M,E)$,

(C.16) $$\boxed{\sigma \in \mathcal{O}(M,E) \Leftrightarrow \bar\partial_E \sigma = 0.}$$

More generally we get

$$\bar{\partial}_E : A^{0,q}(M,E) \to A^{0,q+1}(M,E)$$

and a notion of Dolbeault cohomology H

$$\text{(C.17)} \quad \boxed{H^q_{\bar{\partial}}(M,E) := \underbrace{H^q}_{\ker \bar{\partial}_E^{(q)} / \operatorname{im} \bar{\partial}_E^{(q-1)}} \{ \underbrace{A^{0,\bullet}(M,E)}_{\text{complex}} , \underbrace{\bar{\partial}_E}_{\text{differential}} \}}$$

(cohomology of complex at qth place)

which satisfies the Poincaré lemma $H^q_{\bar{\partial}}(\text{polydisk}, E) = \{0\}$ essentially by Theorem C.2.

Ex / prove a version of $\bar{\partial}$-Poincaré for 1-forms on a punctured disk, using Laurent expansions (to generalize part (ii) of Step 4 in Pf. of Thm. C.2).

Ex / Using a covering by 2 neighborhoods $\mathbb{P}^1 \setminus \{\infty\}$, $\mathbb{P}^1 \setminus \{0\}$, show that $H^{0,1}_{\bar{\partial}}(\mathbb{P}^1, \mathbb{C}) = \{0\}$. (Riemann sphere) (You'll again need to use Laurent expansions at some pt., but this is independent of the last problem).

Ex / Show that $H^{0,1}_{\bar{\partial}}(\mathbb{C}/\Lambda, \mathbb{C})$ is generated by $d\bar{z}$. (cx. torus, $\Lambda = \mathbb{Z}\langle 1, \tau \rangle$, $\tau \in \mathfrak{h}$)

(Hint: suppose $d\bar{z} = \bar{\partial}\beta$ and consider (holomorphic)-translation invariance of $d\bar{z}$.]