E. Kähler manifolds

A Riemannian metric on a smooth manifold M is an everywhere positive-definite section $g \in C^{\infty}(M, \mathrm{Sym}^2 T_M^{\vee})$; the pair (M, g) is called a Riemannian manifold.

In the surface case, this was (up to scaling) essentially the same as an A.C.S.; in general, while this isn't so, it remains of interest to look at Riemannian metrics compatible with an A.C.S. This leads to the notion of a Hermitian manifold and, after imposing another condition, a Kähler manifold. The first time you see it, the Riemannian ↔ Hermitian business can be confusing, so it's best to do it first "on a single tangent plane".

More Linear Algebra

Let $V = \mathbb{C}$-vector space with basis $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$

$W := V^{\mathbb{R}} = \mathbb{R}$-vector space with basis $\{\underbrace{\partial/\partial x_1, \ldots, \partial/\partial x_n}, \underbrace{\partial/\partial y_1, \ldots, \partial/\partial y_n}\}$ e'

$\partial/\partial y_j \xrightarrow{-J} \partial/\partial x_j$, $\quad \partial/\partial x_j \xrightarrow{J} \partial/\partial y_j$

we have $(W, J) \cong (V, M_{\sqrt{-1}})$.

Write $\{dx_j\}$; $\{dx_j, dy_j\}$ for the dual bases; as usual, for $W_{\mathbb{C}}$ resp. $W_{\mathbb{C}}^{\vee}$ we have the bases $\{\partial/\partial z_j, \partial/\partial \bar{z}_j\}$ resp. $\{dz_j, d\bar{z}_j\}$.

e''

Theorem 1: TFAE:

(i) a (real) symmetric bilinear form g on W compatible with J

(ii) a (real) alternating bilinear form ω on W compatible with J

(iii) an Hermitian form h on V

Re

$-\mathrm{Im}$

Also, they are all nondegenerate (or not) together; and (i) & (iii) are positive-definite (or not) together, in which case we also write "$\omega > 0$". $\square$

Proof: Start with (iii); since h is $\begin{cases}\text{linear in } 1^{st} \text{ entry} \\ \text{conj. linear in } 2^{nd} \text{ entry}\end{cases}$ & $h(u,v) = \overline{h(v,u)}$, as an element of $(W^{\otimes 2})^{\vee} \cong (W^{\vee})^{\otimes 2}$

$$h = \underbrace{\sum h_{jk}\, dx_j \otimes dx_k}_{h(\partial/\partial x_j, \partial/\partial x_k) =:} + i \underbrace{\sum h_{jk}\, dy_j \otimes dx_k}_{h(\partial/\partial y_j, \partial/\partial x_k) = h(J\partial/\partial x_j, \partial/\partial x_k) = i h_{jk}} \underbrace{- i \sum h_{jk}\, dx_j \otimes dy_k}_{h(\partial/\partial x_j, \partial/\partial y_k) = h(\partial/\partial x_j, J\partial/\partial x_k) = -i h_{jk}} + \sum h_{jk}\, dy_j \otimes dy_k$$

where $\boxed{h_{jk} = \overline{h_{kj}}}$ (i.e. $\sqrt{-1}$)

Extending this to $(W_{\mathbb{C}}^{\vee})^{\otimes 2}$, $\boxed{h = \sum h_{jk}\, dz_j \otimes d\bar{z}_k}$ (just collect terms!), so*

$$g := \mathrm{Re}\, h = \frac{1}{2}\Big\{ \sum h_{jk}\, dz_j \otimes d\bar{z}_k + \underbrace{\sum \overline{h_{jk}}\, d\bar{z}_j \otimes dz_k}_{\overline{h_{jk}} = h_{kj};\ \overset{\text{swap indices}}{=} \sum h_{jk}\, d\bar{z}_k \otimes dz_j} \Big\} = \frac{1}{2} \sum h_{jk}\, dz_j\, d\bar{z}_k \in \mathrm{Sym}^2 W^{\vee}.$$

$$\boxed{\omega} := -\mathrm{Im}\, h = \frac{i}{2}\Big\{ \sum h_{jk}\, dz_j \otimes d\bar{z}_k - \sum \overline{h_{jk}}\, d\bar{z}_j \otimes dz_k \Big\}$$

(E.1) $$\boxed{= \frac{i}{2} \sum h_{jk}\, dz_j \wedge d\bar{z}_k \in \Lambda^2 W^{\vee} \cap \Lambda^{1,1} W_{\mathbb{C}}^{\vee}}$$

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* Here $dz_j\, d\bar{z}_k := dz_j \otimes d\bar{z}_k + d\bar{z}_k \otimes dz_j$, $dz_j \wedge d\bar{z}_k = dz_j \otimes d\bar{z}_k - d\bar{z}_k \otimes dz_j$, and we can say g resp. ω are in $\mathrm{Sym}^2 W^{\vee}$ resp. $\Lambda^2 W^{\vee}$ (not. $W_{\mathbb{C}}^{\vee}$) since (by construction) they are real.

Assume $h > 0$ (i.e. $h(u,u) > 0 \; \forall u \in W_{(\mathbb{C})}$).

By Gram-Schmidt, there is a unitary basis, in terms of which

(E.2)
$$\boxed{\begin{aligned} &h = \sum {}^{\backprime}dz_j \otimes {}^{\backprime}d\bar{z}_j \quad , \qquad \omega = \tfrac{i}{2} \sum {}^{\backprime}dz_j \wedge {}^{\backprime}d\bar{z}_j \quad , \\ &g = \tfrac{1}{2} \sum {}^{\backprime}dz_j \, {}^{\backprime}d\bar{z}_j = \tfrac{1}{2} \sum \left({}^{\backprime}dx_j^2 + {}^{\backprime}dy_j^2 \right) \; (> 0) . \end{aligned}}$$

Another way to look at all this is: with respect to the real basis e' we have, writing $\mathcal{H} = \{h_{jk}\} = \mathcal{S} - i\mathcal{A}$ where $\mathcal{S} := \operatorname{Re} \mathcal{H}$, $\mathcal{A} := -\operatorname{Im} \mathcal{H}$, ${}^t\mathcal{H} = \bar{\mathcal{H}}$, ${}^t\mathcal{S} = \mathcal{S}$, ${}^t\mathcal{A} = -\mathcal{A}$,

$$[h]_{e'} = \begin{pmatrix} \mathcal{H} & -i\mathcal{H} \\ i\mathcal{H} & \mathcal{H} \end{pmatrix}, \quad [g]_{e'} = \begin{pmatrix} \mathcal{S} & \mathcal{A} \\ -\mathcal{A} & \mathcal{S} \end{pmatrix}, \quad [\omega]_{e'} = \begin{pmatrix} \mathcal{A} & -\mathcal{S} \\ \mathcal{S} & \mathcal{A} \end{pmatrix}$$

while w.r.t. e'' (non-real, hence can't just take Re/Im of matrices)

$$[h]_{e''} = \begin{pmatrix} 0 & \mathcal{H} \\ 0 & 0 \end{pmatrix}, \quad [g]_{e''} = \frac{1}{2}\begin{pmatrix} 0 & \mathcal{H} \\ \bar{\mathcal{H}} & 0 \end{pmatrix}, \quad [\omega]_{e''} = \frac{i}{2}\begin{pmatrix} 0 & \mathcal{H} \\ -\bar{\mathcal{H}} & 0 \end{pmatrix};$$

either way, it's clear that $h = g - i\omega$, and $\begin{cases} {}^t[J][g][J] = [g] \\ {}^t[J][\omega][J] = [\omega] \end{cases}$.

Going back the other way, let

$g : W \times W \to \mathbb{R}$ be a symmetric bilinear form with the compatibility condition $g(Ju, Jv) = g(u,v)$. Then

$$\omega(u,v) := g(Ju, v) = g(J^2u, Jv) = -g(u, Jv) = -g(Jv, u) = -\omega(v,u)$$

is antisymmetric, i.e. in $\Lambda^2 W^\vee$. Noting that J acts on $\Lambda^{p,q} W^\vee_{\mathbb{C}}$ by i^{p-q}, and

$$\Lambda^2 W^\vee_{\mathbb{C}} = \underbrace{\left(\Lambda^{2,0} W^\vee_{\mathbb{C}} \oplus \Lambda^{0,2} W^\vee_{\mathbb{C}}\right)}_{(-1)\text{-eigenspace}} \oplus \underbrace{\Lambda^{1,1} W^\vee_{\mathbb{C}}}_{(+1)\text{-eigenspace}} \quad , \text{ we have}$$

$\omega(Ju, Jv) = g(J^2u, Jv) = -g(u, Jv) = -\omega(v, u) = \omega(u, v)$

$\Rightarrow$ $\underline{\omega \in \Lambda^{1,1} W_{\mathbb{C}}^{\vee}}$, and we also see the equivalence of this condition and ω's J-invariance. (One could also start with $\omega \in \Lambda^2 W^{\vee} \cap \Lambda^{1,1} W_{\mathbb{C}}^{\vee}$ and set $g(u, v) := \omega(u, Jv)$.)

Taking $h := g - i\omega$ gives a Hermitian form, finishing the job. $\square$

Corollary 1: If (M, J) is a complex manifold, TFAE:

(i) a Riemannian metric g compatible with J

(ii) a positive real $(1,1)$-form $\omega \in \underbrace{A^2_{\mathbb{R}}(M)} \cap A^{1,1}(M) =: A^{1,1}_{\mathbb{R}}(M)$
[i.e. $C^{\infty}(M, \Lambda^2 (T_M^{\mathbb{R}})^{\vee})$. Alternatively, $\overline{\omega} = \omega$.]

(iii) a (C^{∞}) Hermitian metric h on T_M.

Definition 1: Let M be a complex manifold with $\omega \in A^{1,1}_{\mathbb{R}}(M)$ s.t. $\omega > 0$. M is Kähler $\Leftrightarrow$ $d\omega = 0$. (ω is called the Kähler form, and g (or h) the Kähler metric.)

Definition 2: A symplectic manifold is a smooth $2n$-manifold (say) M equipped with a nondegenerate form $\omega \in A^2_{\mathbb{R}}(M)$; $\omega^n := \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}}$ is nowhere zero.

Corollary 2: Every Kähler manifold is symplectic; in fact,

$$\boxed{\frac{\omega^n}{n!} = d\mathrm{vol}(g)} \qquad (E.3)$$

Proof: In the unitary basis at a point p *,

$$\omega|_p = \frac{i}{2} \sum {}' dz_j \wedge d\bar z_j = \sum {}' dx_j \wedge {}' dy_j \implies$$

$$\omega^n|_p = n!\, {}'dx_1 \wedge {}'dy_1 \wedge \cdots \wedge {}'dx_n \wedge {}'dy_n = n!\, dvol\Big(\textstyle\sum_j ({}'dx_j^2 + {}'dy_j^2)\Big).$$ □

Corollary 3: $\left.\begin{array}{l} M \text{ compact Kähler} \\ \dim_{\mathbb{C}}(M) = n \end{array}\right\} \implies H^{2k}_{dR}(M, \mathbb{R}) \neq \{0\},\ k = 0, 1, \ldots, n.$

Proof: Obviously $d\omega = 0 \implies d\omega^k = 0$.

Suppose $\omega^k = d\alpha$, $\alpha \in A^{2k-1}_{\mathbb{R}}(M)$. Then

$$\omega^n = d(\omega^{n-k} \wedge \alpha) \implies$$

$$n! \int_M dvol(g) \underset{\text{Stokes}}{=} \int_{\underset{\text{(since } M \text{ compact)}}{\emptyset}} \omega^{n-k} \wedge \alpha = 0$$ (contradiction) □

Corollary 4: (i) Let $N \overset{\iota}{\hookrightarrow} M$ be a complex submanifold** of a Kähler manifold M. Then N is also Kähler.

(ii) [Wirtinger] Assuming N compact of dim. d,

(E.4) $$\boxed{vol(N) = \frac{1}{d!} \int_N \omega^d}$$

Proof: $\iota^* g$ gives a Riemannian metric on N (clearly >0).
By compatibility of J's, it is clear that $\iota^*\omega \in A^{1,1}_{\mathbb{R}}(N)$ is the (1,1) form associated to $\iota^* g$, and hence is $>0 \implies N$ Kähler.
Done by Corollary 3. □

** i.e. N is equiv. $\left\{\begin{array}{l} \text{the image of a holo. immersion (of a } \mathbb{C}\text{-mfld.)} \\ \quad\text{OR} \\ \text{locally cut out by holo. fcns.} \leftarrow \text{use rank theorem} \\ \quad\text{OR} \\ T_pN \text{ everywhere closed under } J \leftarrow \text{use Newlander-Nirenberg} \end{array}\right.$

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* note: (E.2) can only be arranged, by choice of a holo. coord. system, at a single point — NOT on the whole neighborhood

<u>Definition 3</u> : The <u>singular homology</u> of a smooth manifold M is the homology of the complex of singular chains

$$\to C_{q+1}(M;\mathbb{Z}) \xrightarrow[\partial^{(q+1)}]{} C_q(M;\mathbb{Z}) \xrightarrow[\partial^{(q)}]{} C_{q-1}(M;\mathbb{Z}) \to ,$$

$$\text{(E.5)} \quad \boxed{H_q(M;\mathbb{Z}) := \frac{\ker \partial^{(q)}}{\operatorname{im} \partial^{(q+1)}} = \frac{\text{"cycles"}}{\text{"boundaries"}}} .$$

Here $C_q(M;\mathbb{Z}) := \mathbb{Z}\left\langle C^0(\Delta_q, M)\right\rangle$, and

($\mathbb{Z}$ ↑ free abelian group ; Δ_q ↑ q-simplex $\left([0,1]^{q+1}\cap\{\sum t_i = 1\}\right)$ with facets $\Delta_q^i := \Delta_q \cap \{t_i = 0\}$)

$\partial\phi := \sum_i (-1)^i \phi|_{\Delta_q^i}$. The <u>singular cohomology</u> is just the cohomology of the dual complex. Upon extending coefficients to a field $\mathbb{F}\,(=\mathbb{Q},\mathbb{R},\mathbb{C},\text{ etc.})$, we have $H^q(M;\mathbb{F}) \cong H_q(M;\mathbb{F})^\vee$. □

<u>Corollary 5</u> : Let M be a compact Kähler manifold with compact complex submanifold N. Considering the latter as a topological cycle of (real) dimension $2d$, we have

$$0 \neq [N] \in H_{2d}(M, \underset{\text{or } \mathbb{Q}}{\mathbb{Z}})$$

<u>Proof</u> : If $\underset{\mathbb{Q} \ni}{q}\cdot N = \partial \underset{C_{2d+1}(M)}{\Gamma}$ then

$$q(d!)\operatorname{vol}(N) = q\int_N \omega^d = \int_{\partial\Gamma}\omega^d \underset{\text{Stokes}}{=} \int_\Gamma \cancel{d(\omega^d)}^{\,0} = 0 . \quad \text{※}$$

□

Example 1: The Hopf manifolds (compact, complex, $\mathbb{C}$-dim $= n$)

$$M_n := \frac{\mathbb{C}^n \setminus \{0\}}{\left\langle \underset{\forall m \in \mathbb{Z}}{\underline{z} \sim 2^m \underline{z}} \right\rangle} \underset{\text{diffeo.}}{\cong} S^{2n-1} \times S^1 ,$$

are NOT Kähler for $n \geq 2$, since otherwise we would have

$$\underset{\text{Cor. 5}}{0 \neq} [M_1] \in H_2(M_n) = H_2(S^{2n-1} \times S^1) = \{0\} . \qquad \square$$

Example 2: If $\Lambda \subset \mathbb{C}^n$ is a full (rank $2n$) lattice, then

$\omega = \frac{i}{2} \sum dz_i \wedge d\bar{z}_i$ shows the complex n-torus $\mathbb{C}^n/\Lambda$ as Kähler (metric $h = \sum dz_i \otimes d\bar{z}_i$). Obviously also $\mathbb{C}^n$ (= "affine n-space") is Kähler.

KEY Example 3: (Projective n-space)

$$\mathbb{P}^n := \frac{\mathbb{C}^{n+1} \setminus \{\underline{0}\}}{\left\langle \underset{\forall \lambda \in \mathbb{C}^\times}{\underline{z} \sim \lambda \underline{z}} \right\rangle} \qquad \Big(\ni \underbrace{[z_0 : \cdots : z_n]}_{\text{notation for pts.}} \Big)$$

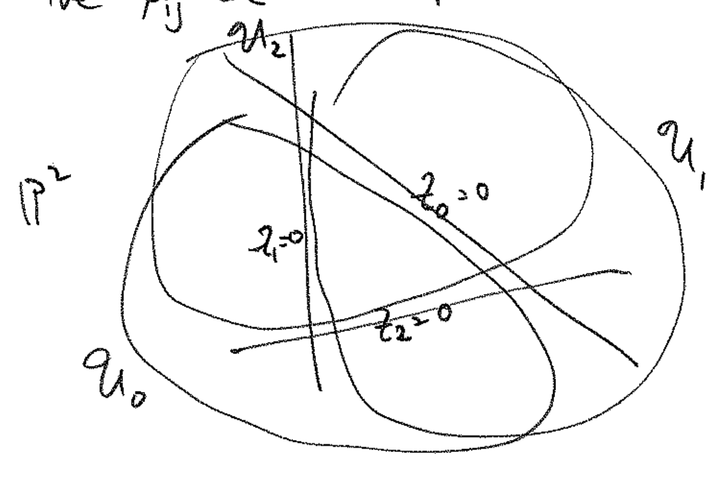
(i) $\mathbb{C}$-manifold structure: Let $\mathcal{U}_i := \{ [\underline{z}] \in \mathbb{P}^n \mid z_i \neq 0 \}$

$$\phi_i : \mathcal{U}_i \xrightarrow{\cong} \mathbb{C}^n$$

$$[\underline{z}] \longmapsto \left(\frac{z_0}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i} \right)$$

(omit)

The ϕ_{ij} are holomorphic where defined, e.g.



$$\phi_{01}(w_1, w_2) = \left(\frac{1}{w_1}, \frac{w_2}{w_1} \right)$$

vert: $\frac{z_0}{z_1}, \frac{z_2}{z_1} \mapsto \frac{z_1}{z_0}, \frac{z_2}{z_0}$

(ii) $\mathbb{C}$-submanifolds : (a) let $F(\underline{z}) \in S^d_{n+1}$ (= homogeneous polynomials of degree d in $(n+1)$ variables: $F(\lambda\underline{z}) = \lambda^d F(\underline{z})$.)

The projective hypersurface*

$$\mathbb{X} = V(F) := \{[\underline{z}] \in \mathbb{P}^n \mid F(\underline{z}) = 0\}$$

is smooth at a point $P \iff$ some $\frac{\partial F}{\partial z_i}(P) \neq 0$ (otherwise, P is a singular point of $\mathbb{X}$). Since $\sum z_i \frac{\partial F}{\partial z_i}(P) \underset{\uparrow}{=} d \cdot F(P) = 0$, another $\frac{\partial F}{\partial z_j}(P) \neq 0$. So if (say)

Ex/Euler formula

$P \in \mathcal{U}_0$, then the local equation $0 = f(\underline{z}) := F(1; \underline{z})$ for $\mathbb{X}$ has nontrivial gradient at P, so the Rank Theorem provides a (holom) chart for a nbhd. of P. Conclude that if $\mathbb{X}$ is smooth (at all points), then $\mathbb{X}$ is a compact complex manifold.

(b) [general projective variety] Given a collection of homogeneous polynomials $F_1, \ldots, F_k$ (of various degrees),

$\mathbb{X} := V(F_1, \ldots, F_k)$ is smooth of codimension** c at $P \iff$

$\exists$ neighborhood $P \in W \subset \mathbb{P}^n$ and subindex set $\{i_1, \ldots, i_c\} \subseteq \{1, \ldots, k\}$ s.t.

$$(E.6) \quad \begin{cases} \mathbb{X} \cap W = V(F_{i_1}, \ldots, F_{i_c}) \cap W \\ \text{AND} \\ \operatorname{rank}\left\{\left(\frac{\partial F_{i_\ell}}{\partial z_j}(P)\right)_{\substack{\ell = 1, \ldots, c \\ j = 0, \ldots, n}}\right\} = c \end{cases}$$

this is not always an easy condition to check!

** or dimension $(n-c)$.

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* I use the notation $\mathbb{X}$ for projective algebraic varieties, which (for present purposes) are vanishing loci $V(\mathfrak{a})$ of homogeneous ideals in S_{n+1} (for some n). Scheme theory gives a more intrinsic characterization (like what we have for manifolds).

"equidimensional"

If X is smooth of the same dimension at each point, the Rank Theorem again endows X with a (compact) cx. manifold structure.

(c) A special case which includes the hypersurface example (a) above, is the case of smooth complete intersections, i.e. where $k = c$ in (b) (no local redundancy of eqns.).

(iii) Fubini–Study metric (or rather, the associated (1,1)-form)

NOTATION:
- $\underline{z} \in \mathbb{C}^{n+1} \rightsquigarrow \|z\|^2 := \sum_j |z_j|^2$
- $\rho_j([\underline{z}]) := \dfrac{\|\underline{z}\|^2}{|z_j|^2} \rightsquigarrow \rho_j \in C^\infty(\mathcal{U}_j)$.
- $\omega_j := \frac{-1}{2\pi i} \partial\bar\partial \log(\rho_j) \in A^{1,1}_{\mathbb{R}}(\mathcal{U}_j)$.

On $\mathcal{U}_{jk}$, $\log(\rho_j) - \log(\rho_k) = \log \frac{z_k}{z_j} + \log \overline{\left(\frac{z_k}{z_j}\right)}$, but $\partial\bar\partial\left(\log \frac{z_k}{z_j} + \log\overline{\left(\frac{z_k}{z_j}\right)}\right) = 0 \implies \underline{\omega_j = \omega_k}$ there.

$\implies$ we have well-defined (1,1) form

$$\text{(E.7)} \qquad \boxed{\omega = \frac{-1}{2\pi i} \partial\bar\partial \log(\|\underline{z}\|^2) \in A^{1,1}_{\mathbb{R}}(\mathbb{P}^n)}$$

with $d\omega = (\partial + \bar\partial)\omega = 0$. It remains to check

POSITIVITY: let A = unitary matrix,

$$\left.\begin{array}{l} \mu_A : \mathbb{P}^n \to \mathbb{P}^n \\ \quad [\underline{z}] \mapsto [A \cdot \underline{z}] \end{array}\right\} \underset{(E.7)}{\implies} \mu_A^* \omega = \omega.$$

The set of such transformations acts transitively on $\mathbb{P}^n$, so it will suffice to check $\omega > 0$ at one point, say $P = [1:0:\cdots:0] \in \mathcal{U}_0$. In coordinates $(w_1, \ldots, w_n)$ we have

$$\rho_0(\underline{w}) = 1 + \sum w_k \bar{w}_k$$

$$\bar{\partial} \log \rho_0(\underline{w}) = \frac{\sum w_k \, d\bar{w}_k}{\rho_0}$$

$$\partial\bar{\partial} \log \rho_0(\underline{w}) = \frac{\sum dw_k \wedge d\bar{w}_k}{\rho_0} - \frac{(\sum \bar{w}_k dw_k) \wedge (\sum w_k d\bar{w}_k)}{\rho_0^2}$$

so

$$\omega|_p = \frac{i}{2\pi} \partial\bar{\partial} \rho_0 \Big|_{(\underline{0})} = \frac{i}{2\pi} \sum dw_k \wedge d\bar{w}_k > 0.$$

Applying Corollaries 3–5, we obtain

<u>Theorem 3</u>: (i) $\mathbb{P}^n$ and all smooth projective varieties are Kähler

(ii) they have nonvanishing even-degree singular + de Rham cohomologies up to twice their dimension.

<u>Remark 1</u>: The function $\log(\rho_j)$ in the above is called a Kähler potential. By the $\partial\bar{\partial}$-lemma (HW #2, Exercise 4), every Kähler metric may be locally described as $\partial\bar{\partial}$ of such a potential.

<u>Remark 2</u>: In fact, the Fubini–Study metric is not just Kähler but <u>Kähler–Einstein</u>: i.e., proportional to the Ricci curvature tensor. These are highly desirable and so difficult to find that numerical methods have come into vogue ("numerical Kähler–Ricci flow").

Here is a more general perspective on Fubini-Study.

Let $M = \mathbb{C}$-manifold

$E \xrightarrow{\pi} M$ = holo. vector bundle (of cx. rank r)

$h = C^\infty$ Hermitian metric on E

Ex/ Using a partition of unity, prove that such an h always exists //

Over $\mathcal{U}_\alpha \subset M$ define a basis of C^∞ sections of E by $\sigma_j^\alpha(p) = \Phi_\alpha^{-1}(p, e_j)$ (where $\Phi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \xrightarrow{\cong} \mathcal{U}_\alpha \times \mathbb{C}^r$ and $e_j = (0, \dots, \overset{j\text{th}}{1}, \dots, 0) \in \mathbb{C}^r$), and $h|_{\mathcal{U}_\alpha}$ is determined by the C^∞ functions $h_{ij}^\alpha(p) := h(\sigma_i^\alpha(p), \sigma_j^\alpha(p))$.

If $r=1$ then $E =: L$ is called a <u>line bundle</u>, and we write $\sigma_1^\alpha =: \sigma_\alpha$, $h_{11}^\alpha =: \rho_\alpha$. We have on $\mathcal{U}_{\alpha\beta}$

$\rho_\beta = h(\sigma_\beta, \sigma_\beta) = h(\Phi_{\alpha\beta}\sigma_\alpha, \Phi_{\alpha\beta}\sigma_\alpha) = |\Phi_{\alpha\beta}|^2 h(\sigma_\alpha, \sigma_\alpha) = |\Phi_{\alpha\beta}|^2 \rho_\alpha$,

where (cf. Defn C-2) the transition functions $\Phi_{\alpha\beta} \in \underset{\text{(nowhere vanishing holomorphic)}}{\mathcal{O}^*}(\mathcal{U}_{\alpha\beta})$.

It follows* that $\partial\bar\partial \log \rho_\alpha = \partial\bar\partial \log \rho_\beta$ on $\mathcal{U}_{\alpha\beta}$, and so

$$(E.8) \qquad \boxed{\omega_{(L,h)} := \left\{ \frac{1}{2\pi i} \partial\bar\partial \log(\rho_\alpha) \right\}_\alpha \in A^{1,1}_{\mathbb{R}}(M)}$$

defines a global real (1,1)-form which is also $d = (\partial + \bar\partial)$-closed.

If $\tilde h$ is another Hermitian metric, with (say) $\tilde h - h$ supported over $\mathcal{U}_\alpha$, then $2\pi i(\tilde\omega - \omega) = \partial\bar\partial \log \frac{\tilde\rho_\alpha}{\rho_\alpha} = d\left(\bar\partial \log \frac{\tilde\rho_\alpha}{\rho_\alpha}\right)$ is exact. (You can't show ω exact in this way,

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$$* \quad \log \rho_\beta = \log \rho_\alpha + \underbrace{\log \Phi_{\alpha\beta}}_{\text{killed by } \bar\partial} + \underbrace{\log \overline{\Phi_{\alpha\beta}}}_{\text{killed by } \partial}$$

because the $\{\bar\partial \log \rho_\alpha\}$ don't "piece together" globally.)

<u>Definition 4</u>: $c_1(L) := [\omega_{(L,h)}] \in H^2_{dR}(M,\mathbb{R})$, which we just checked is independent of h, is called the <u>first Chern class</u> of the line bundle L.

Ex/ show that the $1/\rho_\alpha$ give a Hermitian metric on $L^\vee$ (call this h^*), hence that $c_1(L^\vee) = -c_1(L)$. //

<u>Definition 5</u>: L is <u>positive</u> if there exists an h for which the $(1,1)$ form $\omega_{(L,h)}$ is >0.

<u>Example 4</u>: (i) Define the <u>tautological line bundle</u> on $\mathbb{P}^n$ by

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^n}(-1) := & \{([\underline{z}],v)\in \mathbb{P}^n\times\mathbb{C}^{n+1} \mid v\in\langle \underline{z}\rangle\} & \text{(line in } \mathbb{C}^{n+1} \text{ generated by } \underline{z}) \\ \downarrow \pi & \downarrow & \\ \mathbb{P}^n \ni & [\underline{z}] & \end{array}$$

$$\begin{array}{ccc} \pi^{-1}(\mathcal{U}_i) & \xrightarrow{\Phi_i} & \mathcal{U}_i\times\mathbb{C} \cong \mathbb{C}^{n+1} \\ ([\underline{z}],v) & \longmapsto & (\phi_i([\underline{z}]), v_i) \end{array}$$

$$\Longrightarrow \Phi_{ij}([\underline{z}]) = \frac{v_i}{v_j} = \frac{z_i}{z_j}$$

(ii) Write $\mathcal{O}(1) := \mathcal{O}(-1)^\vee \quad \leftrightarrow \quad \Phi_{ij} = z_j/z_i$

$\mathcal{O}(a) := \mathcal{O}(1)^{\otimes a} \quad \longleftrightarrow \quad \Phi_{ij} = z_j^a/z_i^a$.

Given $P\in S^a_{n+1}$ (homog. poly.), set

$$f_i := \frac{P}{z_i^a} \in \mathcal{O}(\mathcal{U}_i).$$

In $\mathcal{U}_{ij}$, $z_i^a f_i = P = z_j^a f_j$

$$\Longrightarrow f_i = \underbrace{\frac{z_j^a}{z_i^a}}_{\Phi_{ij}} f_j \, . \Rightarrow P \in \mathcal{O}(\mathbb{P}^n, \mathcal{O}(a)) .$$

Indeed, $\mathcal{O}(\mathbb{P}^n, \mathcal{O}(a)) \cong S^a_{n+1}$.

Ex/ $\mathcal{O}(-1)$ has no nontrivial global holomorphic sections. //

(iii) For a Hermitian metric h on $\mathcal{O}(-1)$, we restrict $\sum_{l=0}^n |z_l|^2$ on $\mathbb{C}^{n+1}$ to each fiber $\langle \underline{z} \rangle$. i.e. apply directly to

which yields $\rho_i = 1 + \sum_{l \neq i} \left| \frac{z_l}{z_i} \right|^2$

h on $\Phi_i^{-1}(\phi_i([\underline{z}]), 1) = \left([\underline{z}], \left(\frac{z_0}{z_i}, \dots, \underset{i\text{th place}}{1}, \dots, \frac{z_n}{z_i}\right)\right)$

$$\Rightarrow \omega_{(\mathcal{O}(-1),h)} = \left\{ \frac{1}{2\pi i} \partial \bar{\partial} \log \rho_i \right\}_{i=0}^n = -\omega_{FS} \quad (\text{cf. } (E.7))$$

↑ "Fubini-Study"

$$\parallel$$

$$-\omega_{(\mathcal{O}(1), h^*)}$$

$$\Longrightarrow [\omega_{FS}] = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) .$$

Moreover, if $X \overset{\iota}{\hookrightarrow} \mathbb{P}^n$ is a smooth projective variety, then

$$[\iota^* \omega_{FS}] = c_1(\mathcal{O}_{\mathbb{P}^n}(1)|_X) \text{ and } \underbrace{\iota^* \omega_{FS}}_{\text{positive}} = \omega_{(\mathcal{O}(1)|_X, h^*|_X)} \Longrightarrow$$

($\Rightarrow \mathcal{O}(1)|_X$ is)

Proposition 1: A compact complex manifold M can be a projective variety ONLY IF M admits a positive holomorphic line bundle. □

Example 5: The canonical line bundle on an n-dimensional complex manifold M is the holomorphic bundle

$$K_M := \Lambda^n T_M^{\vee(1,0)}$$

We have $\Omega^n = \mathcal{O}(M, K_M)$ = top degree holomorphic forms.

Ex/ By examining transition functions, show that
$$K_{\mathbb{P}^n} \cong \mathcal{O}(-n-1). \quad //$$

Here is one more beautiful fact about Kähler manifolds.

<u>Proposition 2</u>: Let M be compact Kähler. Then
$$\Omega^q(M) \hookrightarrow H^q_{dR}(M,\mathbb{C}) \qquad \forall q=0,\dots,n.$$

<u>Proof</u>: Let $\{\varphi_1,\dots,\varphi_n\}$ be a <u>local unitary coframe</u>* $(\subset A^{1,0}(\mathcal{U}))$

— these are <u>NOT</u> differentials of holo. coordinates. Say
$$0 \neq \eta = \sum \eta_I \varphi_I \in \Omega^q(M).$$

Then $\eta\wedge\bar{\eta} = \sum \eta_I \bar{\eta}_J \varphi_I \wedge \bar{\varphi}_J$
$$\omega = \frac{i}{2}\sum \varphi_j \wedge \bar{\varphi}_j \implies \omega^{n-q} = \underset{0^*}{C} \sum_{|K|=n-q} \varphi_K \wedge \bar{\varphi}_K$$
$$\int_M \eta\wedge\bar{\eta}\wedge\omega^{n-q} = C\int_M \sum_I |\eta_I|^2 d\mathrm{vol}(g) \neq 0.$$

Can do this b/c M compact

Now <u>suppose $\eta = d\psi$</u>. Then $\begin{cases} d\eta = 0 \Rightarrow d\bar{\eta} = 0 \\ \boxed{\omega \text{ Kähler} \Rightarrow d\omega = 0} \end{cases} \Longrightarrow$
$$\int_M \eta\wedge\bar{\eta}\wedge\omega^{n-q} = \int_M d(\psi\wedge\bar{\eta}\wedge\omega^{n-q}) \underset{\text{Stokes}}{=} 0,$$
<u>contradiction</u>.

Finally, <u>suppose $d\eta \neq 0$</u>. But $d\eta \in \Omega^{q+1}(M)$, and then the above argument shows that $d\eta$ cannot be exact. ※ $\square$

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* for the Kähler metric h

So far we've had the following examples of Kähler manifolds.

- projective space $\mathbb{P}^n$
- smooth projective varieties $\overline{V}(\mathfrak{d}) \subseteq \mathbb{P}^n$, $\mathfrak{d} \subseteq \mathbb{C}[z_0, \dots, z_n] = \bigoplus_d S^d_{n+1}$ a homogeneous prime ideal *
- affine space "$\mathbb{A}^n$" $(= \mathbb{C}^n)$
- smooth affine varieties $V(\mathcal{Q}) \subseteq \mathbb{A}^n$, $\mathcal{Q} \subseteq \mathbb{C}[z_1, \dots, z_n]$ a prime ideal*.
- complex tori
- Riemann surfaces: Riemannian real-2-manifold $\overset{\text{equiv.}}{\longleftrightarrow}$ complex 1-manifold

Here are a couple more.

<u>Example 6</u>: E holo. vector bundle with Hermitian metric h
$\downarrow$
M <u>compact</u> Kähler with form ω_M

$\mathcal{O}_{\mathbb{P}(E)}(-1)$ the tautological bundle
$\downarrow$
$\mathbb{P}(E)$ the fiberwise projectification of E
$\left(\begin{smallmatrix} \downarrow \pi \\ M \end{smallmatrix}\right)$

Restricting h to fibers of $\mathcal{O}_{\mathbb{P}(E)}(-1)$ produces a metric. Let $\omega_E := -$(its Chern form), which pulls back to ω_{FS} (hence is >0) on each fiber; it could still be negative "in the horizontal direction", but this negativity is bounded below since M is compact. So $\omega_E + \lambda \pi^* \omega_M > 0$ for $\lambda > 0$ suff. large. $\square$

---

* in both cases necessarily finitely generated by the Hilbert basis theorem. With this in mind, one can toggle between projective & affine varieties:

<u>ideals</u>: $\mathfrak{d} \mapsto \mathcal{Q}$: divide generating eqn. of deg. d by Z_0^d

$\mathcal{Q} \mapsto \mathfrak{d}$: write $z_i := Z_i / Z_0$ and clear denominators

<u>varieties</u>: $\overline{V} \mapsto V$: intersect $\overline{V}$ with $\mathcal{U}_0 \cong \mathbb{A}^n \subseteq \mathbb{P}^n$

$V \mapsto \overline{V}$: take closure in $\mathbb{P}^n$ (this need not preserve smoothness!)

Example 7: $\mathcal{N}_{N/M} := T_M|_N \big/ T_N$ "normal bundle" (holomorphic)

$$\begin{array}{c} \mathcal{N}_{N/M} \\ \downarrow \\ N \subset M \end{array}$$

(compact $\mathbb{C}$-submanifold) (complex manifold)

There exists a construction called the blow-up of M along N:

$$\begin{array}{c} B_N(M) \\ \downarrow \beta \\ M \end{array} \quad \text{with} \quad \begin{cases} \beta^{-1}(M\setminus N) \cong M \setminus N \\ \beta^{-1}(N) \cong \mathbb{P}(\mathcal{N}_{N/M}) \end{cases}$$

← dimension $= \dim_{\mathbb{C}}(M) - 1$.

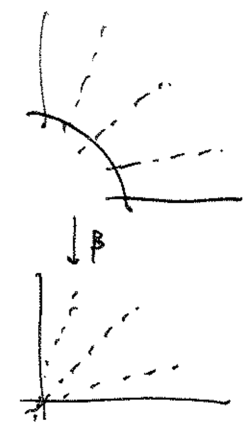
Locally (on $\mathcal{U} \subset M$), the idea is simple (see Voisin for the full details):

if $N \cap \mathcal{U} = \{ \underline{u} \in \mathcal{U} \mid f_i(\underline{u}) = 0 \ \ (i = 1, \dots, k) \} \overset{\text{codim } k}{\subset} \mathcal{U}$

then $\beta^{-1}(\mathcal{U}) = \{ (\underline{u}, [\underline{z}]) \in \mathcal{U} \times \mathbb{P}^{k-1} \mid z_i f_j(\underline{u}) = z_j f_i(u) \ \ (\forall\, i, j = 1, \dots, k) \}$.

To get anything, you need $k \geq 2$. The prototypical example is:

$$\begin{array}{c} B_{(0,0)}(\mathbb{C}^2) = \{ ((w_0, w_1), [\underline{z}]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid z_0 w_1 = z_1 w_0 \} \\ \downarrow \\ \mathbb{C}^2 \end{array}$$



where $(0,0)$ has been "blown up" into a $\mathbb{P}^1$, separating lines thru the origin by slope — useful if you want to make a meromorphic fcn. like w_1/w_0 well-defined.

Terminology: going $M \rightsquigarrow B_N(M)$ is "blowing up (along) N"

going $B_N(M) \rightsquigarrow M$ is "blowing down $\beta^{-1}(N)$"

Blowing up is always possible and preserves the Kähler property; blowing down (a given submanifold* to something of lower dimension) is only possible under special circumstances and need not preserve the Kähler property even when it's possible. $\square$

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* not starting from the assumption that it is $\beta^{-1}(N)$

We now turn to a discussion of the 3 conditions which are equivalent to the Kähler condition $d\omega = 0$. To this end we first pass back to the general situation of a smooth vector bundle E over a smooth manifold M.

Definition 6: (i) A connection on E is an $\mathbb{R}$-linear map

$$\nabla : C^\infty(M,E) \longrightarrow C^\infty(M, T_M^\vee \otimes E) =: A^1(M,E)$$

s.t. $\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma \quad (\forall f \in C^\infty(M))$.

This gives rise to directional derivatives (in direction $\eta \in C^\infty(M, T_M)$)

$$\nabla_\eta : C^\infty(M,E) \longrightarrow C^\infty(M,E)$$
$$\sigma \longmapsto (\nabla\sigma)(\eta).$$

(ii) If $\sigma_1, \dots, \sigma_r$ is a basis of sections (over $\mathcal{U} \subset M$), the matrix of connection 1-forms is defined by

$$\nabla\sigma_i = \sum_k \theta_i^k \otimes \sigma_k \quad , \quad \theta_i^k \in A^1(\mathcal{U}).$$

(iii) A section σ is flat if $\nabla\sigma = 0$.

Example 8: The Levi-Civita connection on the tangent bundle of a Riemannian manifold (M,g).

Let $E = T_M$, with local coords. $\{x_i\}$ on $\mathcal{U}$ (and hence local sections $\{\partial_{x_i} =: \partial_i\}$ of T_M).

Define the Christoffel symbols by

$$\theta_i^k =: \sum \Gamma_{ij}^k dx_j$$

and the torsion of the connection by

$$T_{ij}^k := \Gamma_{ij}^k - \Gamma_{ji}^k .$$

Definition 7: The Levi-Civita connection on T_M is the unique connection with $T=0$ ($\underset{Ex}{\Leftrightarrow}$ $\nabla_X Y - \nabla_Y X = [X,Y]$ $\forall X,Y$) which is compatible with g:

$$d(g(X,Y)) = g(\nabla X, Y) + g(X, \nabla Y). \quad //$$

To see that it exists and is unique, derive the formula: if $\begin{cases} X = \partial_k \\ Y = \partial_j \end{cases}$

$$d(\underbrace{g(\partial_k, \partial_j)}_{g_{kj}})\partial_i = g(\underbrace{\nabla_{\partial_i}\partial_k}_{\sum \Gamma^l_{ki}\partial_l}, \partial_j) + g(\partial_k, \underbrace{\nabla_{\partial_i}\partial_j}_{\sum \Gamma^l_{ji}\partial_l})$$

$$\frac{\partial g_{kj}}{\partial x_i} = \sum_l g_{lj}\Gamma^l_{ki} + \sum g_{kl}\Gamma^l_{ji}$$

(E.9) $$\frac{\partial g_{kj}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ji}}{\partial x_k} = \sum_l \left(\cancel{g_{lj}\Gamma^l_{ki}} + g_{kl}\Gamma^l_{ji} + g_{lk}\Gamma^l_{ij} + \cancel{g_{il}\Gamma^l_{kj}} - \cancel{g_{li}\Gamma^l_{jk}} - \cancel{g_{jl}\Gamma^l_{ik}} \right)$$

$$\underset{(T=0)}{=} 2\sum g_{kl}\Gamma^l_{ji}$$

Multiplying both sides by the inverse matrix g^{pk} and summing over k, we get

(E.10) $$\boxed{\Gamma^p_{ji} = \frac{1}{2}\sum_k g^{pk} \times \mathrm{LHS(E.9)}}$$

which is visibly symmetric in i & j.

Example 9: The Chern connection on a holomorphic Hermitian vector bundle (E,h) over a $\mathbb{C}$-manifold M. [cf pp. 22–23 for defn. of $\bar\partial_E$]

We have to refine defn. 6 a bit: start with a $\mathbb{C}$-linear

$$\nabla : C^\infty(M,E) \to C^\infty(M, (T^\vee_M \otimes_{\mathbb{R}} \mathbb{C}) \otimes E) = A^{1,0}(M,E) \oplus A^{0,1}(M,E)$$

$$\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$$

Taking $\sigma_1, \ldots, \sigma_r$ to be holomorphic sections, write

$$\left.\begin{aligned} \nabla^{(1,0)} \sigma_i &=: \sum_k \theta_i^k \otimes \sigma_k \\ \nabla^{(0,1)} \sigma_i &=: \sum_k \tau_i^k \otimes \sigma_k \end{aligned}\right\} \quad \text{where } \theta_i^k \in A^{1,0}(\mathfrak{U}) \text{ and } \tau_i^k \in A^{0,1}(\mathfrak{U}),$$

and notice $\nabla^{(0,1)} = \bar\partial_E \iff \tau = 0$.

Definition 8: The Chern connection on (E,h) is the unique (complex) connection with $\nabla^{(0,1)} = \bar\partial_E$ which is compatible with h:

$$d\,(h(\chi,\psi)) = h(\nabla\chi,\psi) + h(\chi,\nabla\psi).$$

//

Again, we derive the formula by taking $\chi = \sigma_i$, $\psi = \sigma_j$ and noticing that

$$\partial\,(h(\chi,\psi)) = h(\nabla^{1,0}\chi,\psi) + h(\chi, \underbrace{\nabla^{0,1}\psi}_{=\bar\partial\psi = \bar\partial(\sigma_j) = 0})$$

assume — since σ_j is a holo. section.

So letting $\partial/\partial z_\ell$ operate on both sides, and writing $\theta_i^k =: \sum \mu_{ij}^k \, dz_j$,

$$\frac{\partial h_{ij}}{\partial z_\ell} = h\Big(\underbrace{\nabla^{1,0}_{\partial/\partial z_\ell} \sigma_i}_{=\sum_k \mu^k_{i\ell}\sigma_k}, \sigma_j\Big) = \sum_k h_{kj}\,\mu^k_{i\ell} \quad \Longrightarrow$$

(E.11)
$$\boxed{\mu^p_{i\ell} = \sum_j h^{jp} \frac{\partial h_{ij}}{\partial z_\ell}}.$$

Now let $E = T_M$ (holomorphic tangent bundle), with Hermitian metric h, & $g := \mathrm{Re}(h)$, $\omega := -\mathrm{Im}(h)$. We can consider BOTH the Levi-Civita ∇ for $(T_M^{\mathbb{R}}, g)$ and Chern ∇ for (T_M, h), where in the latter case the $\sigma_i = \partial/\partial z_i$ come from the choice of local coordinates.

<u>Theorem 4</u> : TFAE :

(i) $d\omega = 0$ (h is Kähler)

(ii) h osculates to order 2 the Euclidean metric everywhere

(iii) $\nabla_{LC} = \nabla_{Chern}$

(iv) J commutes with ∇_{LC}.

<u>Proof</u>: (i) $\Rightarrow$ (ii). At a point p, we can write holomorphic coordinates s.t. $\omega = \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j$ <u>at that point</u>. More precisely,

$$\omega = \frac{i}{2}\sum_{i,j,k}\left(\delta_{ij} + a_{ijk}z_k + a_{ij\bar{k}}\bar{z}_k + \underbrace{O(|z|^2)\big)\, dz_i \wedge d\bar{z}_j}_{\text{henceforth called ``[2]''}}\right.$$

Now

$$\omega = \bar{\omega} = \frac{i}{2}\sum\left(\delta_{ij} + \overline{a_{ijk}}\,\bar{z}_k + \overline{a_{ij\bar{k}}}\,z_k\right) dz_j \wedge d\bar{z}_i + [2]$$

$\Rightarrow$

(E.12) (relabel $i \leftrightarrow j$) $\boxed{a_{ji\bar{k}} = \overline{a_{ijk}}}$.

Moreover,

$$0 = d\omega = \frac{i}{2}\sum\left(a_{ijk}\, dz_k \wedge dz_i \wedge d\bar{z}_j + a_{ij\bar{k}}\, dz_i \wedge d\bar{z}_j \wedge d\bar{z}_k\right) + [1]$$

$\Rightarrow$

(E.13) $\boxed{a_{ijk} = a_{kji}}$

Defining coordinates $\{w_k\}$ by

$$z_k = w_k - \frac{1}{2}\sum_{l,m} a_{mkl}\, w_l w_m ,$$

we have by (E.13) $dz_k = dw_k - \sum a_{mkl}\, w_l\, dw_m$. So

$$\frac{2}{i}\omega = \sum_i \Big(dw_i - \sum_{m,l} a_{mil}\, w_l\, dw_m\Big)\wedge\Big(d\bar w_i - \sum_{p,q}\overline{a_{piq}}\,\bar w_q\, d\bar w_p\Big)$$

$$+ \sum_{i,j,k}\big(a_{ijk} w_k + a_{ij\bar k}\bar w_k\big)\, dw_i\wedge d\bar w_j \quad + \quad [2]$$

$$= \sum_{i,j}\Big(\delta_{ij} + \sum_k \big(\cancel{a_{ijk} w_k} + \underbrace{a_{ij\bar k}\bar w_k}\; - \cancel{a_{ijk} w_k} - \underbrace{\overline{a_{jik}}\,\bar w_k}\big)\Big)\, dw_i\wedge d\bar w_j$$

$$+ [2]$$

Cancel by (E.12)

$$= \sum_j dw_j \wedge d\bar w_j \quad + \quad [2].$$

□

(ii) $\Rightarrow$ (iii). In the coordinates $\{w_i\}$ above, $h_{ij}(p) = g_{ij}(p) = \delta_{ij}$ and the 1^{st} partials of the h_{ij} resp. g_{ij} at p are all zero. Hence, at p, $\mu^i_{jk}(p) = 0 = \Gamma^i_{jk}(p) \Longrightarrow (\nabla(\partial/\partial z_i))(p) = 0$ for both ∇_{LC} and ∇_{Chern}. Since the last statement does not depend on the choice of local coordinates (by uniqueness of the connections ∇_{LC} & ∇_{Chern}), it is true at all points.

□

(iii) $\Rightarrow$ (iv). $(\nabla_{LC} =)\ \nabla_{Chern}$ is $\mathbb{C}$-linear, hence commutes with J.

□

(iv) $\Rightarrow$ (i). Let $\nabla = \nabla_{LC}$. Then for ξ, χ vector fields

$$d(g(\xi,\chi)) = g(\nabla\xi,\chi) + g(\xi,\nabla\chi)$$

$\Downarrow$ ($\omega(\cdot,\cdot) = g(J(\cdot),(\cdot))$ and $[\nabla, J] = 0$)

$$d(\omega(\xi,\chi)) = \omega(\nabla\xi,\chi) + \omega(\xi,\nabla\chi)$$

$\Downarrow$ (apply another vector field η)

$$\boxed{\eta(\omega(\xi,\chi)) = \omega(\nabla_\eta \xi,\chi) + \omega(\xi,\nabla_\eta\chi)} \tag{E.14}$$

By the HW Exercise on p. 6 (§I.B),

$$d\omega(\eta,\xi,\chi) = \boxed{\eta(\omega(\xi,\chi))} + \xi(\omega(\chi,\eta)) + \chi(\omega(\eta,\xi)) - \omega([\eta,\xi],\chi) - \omega([\xi,\chi],\eta) - \omega([\chi,\eta],\xi)$$

using $T=0$ + ω's antisymmetry

$$= \boxed{-\omega(\nabla_\eta \xi,\chi)} - \omega(\chi,\nabla_\xi \eta) - \omega(\nabla_\xi \chi,\eta) - \omega(\eta,\nabla_\chi \xi) - \omega(\nabla_\chi \eta,\xi) \boxed{-\omega(\xi,\nabla_\eta \chi)}$$

$= 0$ after applying (E.14) and its two cyclic permutations.

□

Final remarks on how the Kähler property sits in the pantheon of complex manifolds.

Let $M = \mathbb{C}$-manifold.

Definition 9: M is Stein $\iff$ $\exists$ holo. embedding $h: M \hookrightarrow \mathbb{C}^N$ s.t. $h(M)$ is closed in $\mathbb{C}^N$. //

Remarks on Stein manifolds M: (i) By Corollary 4(i), M is Kähler

(ii) Maximum modulus principle $\Longrightarrow$ M noncompact

(iii) Smooth affine varieties are Stein.

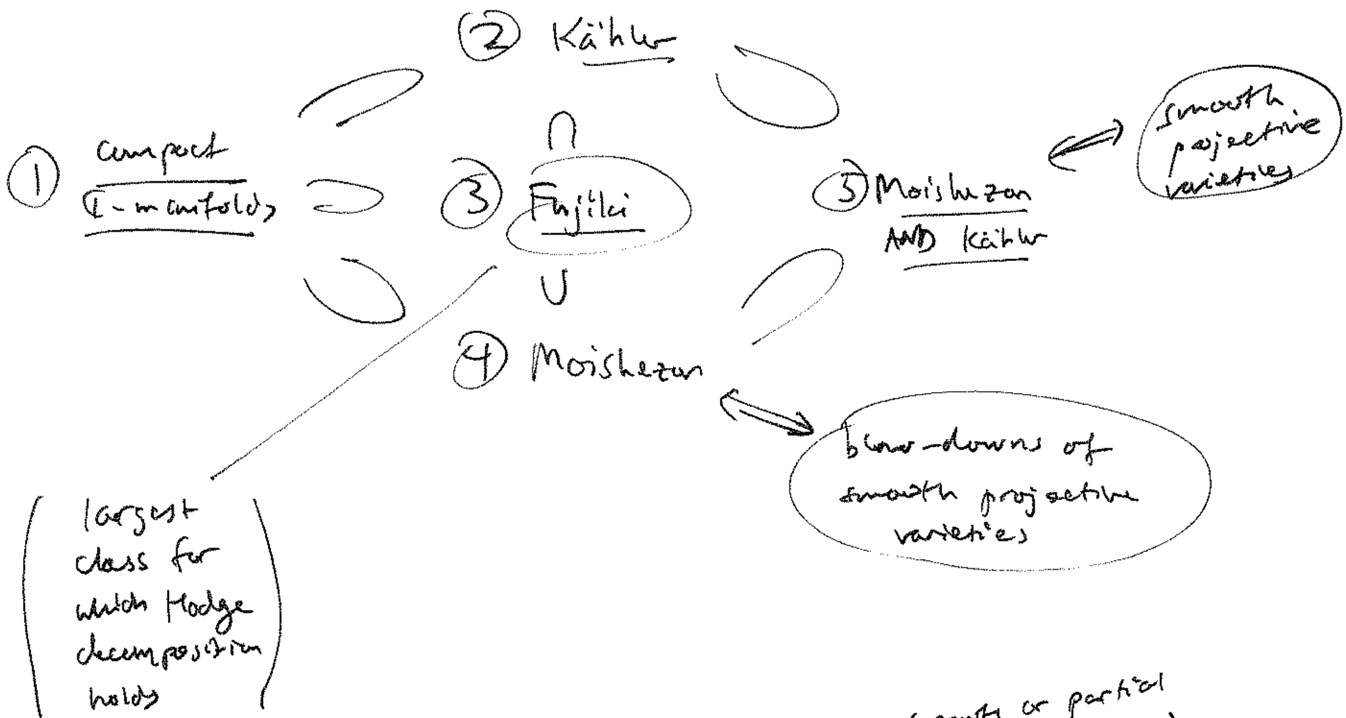
Now restrict to M compact $\mathbb{C}$-manifold.

field of meromorphic fcns.

Definition 10: (i) M is Moishezon $\iff \operatorname{trdeg}_{\mathbb{C}} \operatorname{Mer}(M) = \dim_{\mathbb{C}}(M)$

(ii) M is Fujiki $\iff$ blow-down of Kähler. //

We then have the nice picture



We also note the fundamental results (proofs or partial proofs later on)

Chow's theorem: $M \subset \mathbb{P}^N$ complex submanifold $\iff$ M (smooth) projective variety

$\underbrace{(\text{"global analytic} \Rightarrow \text{global algebraic"})}_{\text{GAGA}}$

Kodaira Embedding Theorem: M admits positive holo. line bundle $\iff$ M is projective.

Examples: (a) 5 ∋ all compact Riemann surfaces

(b) $1 \setminus 2$ ∋ Hopf manifolds

(c) $4 \setminus 5$ ∋ Some Calabi-Yau 3-folds studied by physicists

(d) $2 \setminus 5$ ∋ "most" complex tori (the ones w/o a positive line bundle).

Those complex tori in 5 (admitting a projective embedding) are called Abelian varieties. //

We will have more to say about (a) & (d) later.