F. Sheaves + Cohomology

PRELIMINARIES

Definition 1: Let $\{A_i\}_{i\in I = \text{directed set}^{\circledast}}$ be a direct system of abelian groups, i.e. a family with morphisms

$$f_{ij} : A_i \to A_j \quad (\forall \text{ pairs } i \le j) \quad \text{satisfying} \quad \begin{cases} f_{ii} = \mathrm{id}_{A_i} & \forall i \\ f_{ik} = f_{jk} \circ f_{ij} & \forall i \le j \le k. \end{cases}$$

Then the direct limit is an abelian group defined by

$$\varinjlim_{i\in I} A_i := \coprod_{i \in I} A_i \Big/ \left\langle x_i^{\in A_i} \sim x_j^{\in A_j} \Leftrightarrow \begin{matrix} f_{ik}(x_i) = f_{jk}(x_j) \\ \text{for some } k (\ge i, j) \end{matrix} \right\rangle . \qquad \square$$

The most illuminating way to think of elements is as "strings of elements starting at any A_i" and which are equal if they "eventually converge"!

Example 1: (i) Let ω_q = group of q^{th} roots of 1 (in $\mathbb{C}^*$)

p a prime

then $\varinjlim_{n\in\mathbb{N}} \omega_{p^n}$ = group of roots of 1 of order some power of p

(ii) $\varinjlim_{\substack{U \ni z_0 \\ U\subset\mathbb{C} \\ \text{open}}} \mathcal{O}(U)$ = germs of analytic fns. at z_0

(= convergent power series) $\square$

Remark 1: (a) Direct limits preserve exactness (Exercise?)

(b) A subset $J \subset I$ is cofinal if "$\exists j \ge i\ (\forall i)$". One then has $\varinjlim_{i\in I} A_i = \varinjlim_{j\in J} A_j$.

$\circledast$ a poordered set with upper bounds for each pair of elements

("$\le$" is reflexive + transitive)

<u>Definition 2</u>: (i) A <u>category</u> $\mathcal{C}$ consists of

(a) $Ob(\mathcal{C})$ = set of objects

(b) for each $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ = set of <u>morphisms</u> (incl. <u>id_X</u> $\forall X \in \mathcal{C}$)

(c) <u>compositions</u> of morphisms $\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$ (satisfying associativity).

(ii) A <u>functor</u> $F : \mathcal{C} \to \mathcal{D}$ between categories is

(a) a map $Ob(\mathcal{C}) \to Ob(\mathcal{D})$ [associating to each $X \in Ob(\mathcal{C})$ a unique $F(X) \in Ob(\mathcal{D})$]

(b) for each $X, Y \in \mathcal{C}$, a map

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \begin{cases} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) & \text{[for \underline{covariant} functor]} \\ \text{or} \\ \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X)) & \text{[for \underline{contravariant} functor]} \end{cases}$$

(c) respecting identities & compositions.

(iii) A <u>natural transformation</u> $\eta : F \to G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is a collection of morphisms $\eta_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ (one for each $X \in \mathcal{C}$) such that the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \eta_X \downarrow & & \downarrow \eta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

commutes. □

<u>Example 2</u>: Let <u>Ab</u> = category of abelian groups with usual homomorphisms, and let $A, B \in$ <u>Ab</u> and $\eta \in \mathrm{Hom}_{Ab}(A, B)$ be given. Then

$$\mathrm{Hom}_{Ab}(-, A), \mathrm{Hom}_{Ab}(-, B) : Ab \to Ab$$

are <u>contravariant</u> functors and η gives a natural transformation between them. (Ex/ work this out) □

DEFINITIONS

Now let M = smooth manifold

$Op(M)$ = category of open sets on M with inclusion morphisms:

$$\mathrm{Hom}_{Op(M)}(V, U) = \begin{cases} \emptyset, & V \not\subset U \\ \underbrace{\{\iota_{VU} : V \hookrightarrow U\}}, & V \subset U \end{cases}$$

(1-element set consisting of the inclusion.)

<u>Definition 3</u>: (i) A <u>presheaf</u> of abelian groups* on M is a contravariant functor

$$\mathcal{F} : Op(M) \to Ab .$$

(writing $\mathcal{F}(\iota_{VU}) =: \rho^{(\mathcal{F})}_{VU} : \mathcal{F}(U) \to \mathcal{F}(V)$, the condition (ii)(c) in Defn 2 $\Rightarrow$ $\rho_{VW} = \rho_{VU} \circ \rho_{UW}$ and $\rho_{UU} = id_{\mathcal{F}(U)}$.)

(ii) A presheaf $\mathcal{F}$ is a <u>sheaf</u> $\overset{\text{def}}{\iff}$ for each $U \in Op(M)$ and each covering $\{V_j\} (\subset Op(M))$ of U, we have

(a) [vanishing] $f \in \mathcal{F}(U)$ and $\rho_{V_j U}(f) = 0$ $(\forall j)$ $\Rightarrow$ $f = 0$

(b) [glueing] $\{f_j \in \mathcal{F}(V_j)\}$ and $\rho_{V_i \cap V_j, V_j}(f_j) = \rho_{V_i \cap V_j, V_i}(f_i)$ $(\forall i, j)$
$\Rightarrow \exists f \in \mathcal{F}(U)$ s.t. $f_j = \rho_{V_j U}(f)$ $(\forall j)$.

(iii) A <u>morphism</u> $\phi : \mathcal{F} \to \mathcal{G}$ of (pre)sheaves is a natural transformation.

(That is, a collection $\{\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)\}_{U \in Op(M)}$ s.t. $\rho^{\mathcal{G}}_{VU} \circ \phi_U = \phi_V \circ \rho^{\mathcal{F}}_{VU}$.)

$\square$

<u>Remark 2</u>: The $\overbrace{\text{kernel}}^{\ker \phi}$ of a morphism of sheaves, defined by $(\ker \phi)(U) := \ker(\phi_U)$, is a sheaf (by using (ii)(a) for $\mathcal{F}$ & $\mathcal{G}$).

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* this can be adapted to <u>Gp</u>, <u>Ring</u>, <u>R-mod</u>, etc. — not just <u>Ab</u>.
(fixed ring)

Let $Sh(M)$ & $PreSh(M)$ denote the resulting categories.
For $\mathcal{F} \in$ either, and $p \in M$, we have the

Definition 4: (i) The stalk of $\mathcal{F}$ at p is

$$\mathcal{F}_p := \varinjlim_{\substack{U \ni p \\ (U \in Op(M))}} \mathcal{F}(U).$$

(Note that by Remark 1(ii), it suffices to consider "balls" about p.)

(ii) A morphism $\phi : \mathcal{F} \to \mathcal{G}$ is injective (resp. surjective) $\overset{\text{def.}}{\iff}$ the morphisms $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ (in $\underline{Ab}$) are, for each p.

In particular, a complex of sheaves is exact (at some term) $\iff$ it is exact at the stalk level.

Now set $C(U)$ (any open set $U \subset M$) := set of open covers of U, with typical element $\mathcal{V} = \{V_\alpha\}$.

Given $\mathcal{F} \in PreSh(M)$, set

- $\mathcal{F}'(U) := \mathcal{F}(U) \Big/ \{\sigma \in \mathcal{F}(U) \mid \rho_{V_\alpha U}(\sigma) \overset{(\forall \alpha)}{=} 0 \text{ for some } \mathcal{V} \in C(U)\}$
- $\mathcal{F}'(\mathcal{V}) := \{(\sigma_\alpha \in \mathcal{F}'(V_\alpha)) \mid \sigma_\alpha|_{V_{\alpha\beta}} = \sigma_\beta|_{V_{\alpha\beta}} \ (\forall \alpha, \beta)\}$ (collections)
- $(\Phi\mathcal{F})(U) := \varinjlim_{\mathcal{V} \in C(U)} \mathcal{F}'(\mathcal{V})$ (ordered by refinement)

(which satisfies $\Phi\mathcal{F} = \mathcal{F}$ if $\mathcal{F}$ was already a sheaf). This defines the sheafification functor

$$\Phi : PreSh(M) \to Sh(M).$$

Remark 3: (i) Φ preserves stalks

(ii) If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then we set $(\widetilde{\mathrm{Im}}\,\phi)(U) := \mathrm{Im}(\phi_U)$, $\mathrm{Im}\,\phi := \Phi\,\widetilde{\mathrm{Im}}\,\phi$. If ϕ surjects* onto a subsheaf $\mathcal{G}' \subset \mathcal{G}$, then $\mathrm{Im}\,\phi = \mathcal{G}'$ (due to the $\cong$ on the stalk level) — cf. [Voisin] for the proof. //

<u>Remark 4</u>: In [Ahlfors] and other books, you may have come across the definition of a sheaf as a topological space $\mathcal{S}$ with C^0 map $\pi : \mathcal{S} \to M$ which is a local homeomorphism, and s.t. $\left\{\begin{array}{l}\underbrace{\pi^{-1}(p)}_{\text{stalk}} \in Ab \text{ (or whatever) for each } p, \text{ and} \\ \text{stalkwise addition is } C^0 \text{ from } \mathcal{S}\times\mathcal{S} \to \mathcal{S}.\end{array}\right.$

The proof that this is equivalent to the above definition may be found in [Lewis] (Ch. 3 appendix). //

$\boxed{\text{EXAMPLES}}$

<u>Example 3</u>: Let $\mathcal{F}$ be the presheaf $\mathcal{F}(U) = \mathbb{Z}$ $(\forall U)$.

Then $\mathcal{F}' = \mathcal{F}$, but $(\Phi\mathcal{F})(U) = \mathbb{Z}^{\pi_0(U)}$. The resulting sheaf is written $\underline{\mathbb{Z}}$ or $\mathbb{Z}_M$; there are obvious other "constant sheaves" $\underline{\mathbb{Q}}$, $\underline{\mathbb{R}}$, $\underline{\mathbb{C}}$, etc.

<u>Example 4</u>:

<u>sheaves of functions</u>: C^∞, $\mathcal{O}$, $\mathcal{O}^*$, $\mathcal{M}$ — mero. fcns.

<u>sheaves of forms</u>: A^k, $A^{p,q}$, Ω^p

<u>sheaf of holo. vector fields</u>: Θ^1

only defined on a <u>complex</u> manifold

<u>sheaves of $\left\{\begin{array}{l} C^\infty \\ \text{holo.}\end{array}\right.$ sections of a vector bundle E (of rank r)</u> : $C^\infty(E)$, $\mathcal{O}(E)$.

(Notation: $\mathcal{O}(E)(U) = \mathcal{O}(U,E)$.) [Voisin] explains in detail that this is the same as a sheaf locally $\cong$ to $\mathcal{O}_M^{\oplus r}$ resp. $(C_M^{\infty})^{\oplus r}$, i.e. a <u>locally free sheaf of rank r</u>.

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* i.e. ϕ_p is surjective $(\forall p \in M)$ onto $\mathcal{G}'_p$

<u>Example 5</u> : Say $\mathcal{F}(U) = \mathcal{O}(U)/\underline{\mathbb{Z}}(U)$ on $\mathbb{C}$-mfld. Then $\mathcal{F}' = \mathcal{F}$, but

$$\Phi\mathcal{F}(U) = \{\text{multivalued holo. fcns. on } U \text{ with monodromy} \in \mathbb{Z}\}$$

$$\cong \Big\downarrow \exp(2\pi i(\cdot))$$

$$\mathcal{O}^*(U)$$

"exponential sequence"

and $0 \to \underline{\mathbb{Z}} \to \mathcal{O} \xrightarrow{e^{2\pi i(\cdot)}} \mathcal{O}^* \to 0$

is a short-exact sequence of sheaves.

<u>Example 6</u> : By the Poincaré lemmas, one has the long exact sequences

$$0 \to \boxed{\mathbb{C}} \to \underset{(A^0)}{C^\infty} \xrightarrow{d} A^1 \xrightarrow{d} A^2 \to \cdots \qquad \text{on } C^\infty \text{ mfld.}$$

and

$$0 \to \boxed{\mathcal{O}} \to \underset{(A^{0,0})}{C^\infty} \xrightarrow{\bar\partial} A^{0,1} \xrightarrow{\bar\partial} A^{0,2} \to \cdots \qquad \text{on } \mathbb{C}\text{-mfld.} ;$$

and it's an easy exercise to prove that

$$0 \to \boxed{\mathbb{C}} \to \underset{(\Omega^0)}{\mathcal{O}} \xrightarrow{d(=\partial)} \Omega^1 \xrightarrow{d(=\partial)} \Omega^2 \to \cdots \qquad (\text{also on } \mathbb{C}\text{-mfld})$$

is one too. Typically one calls these <u>resolutions</u> of the boxed term: resolution of

$\mathbb{C}$ by $(A^\bullet_M, d)$ <u>de Rham</u>

$\mathcal{O}$ by $(A^{0,\bullet}_M, \bar\partial)$ <u>Dolbeault</u>

$\mathbb{C}$ by $(\Omega^\bullet_M, d)$ <u>holomorphic de Rham</u>.

<u>Example 7</u> : An important source of examples are the <u>image functors</u> for sheaves: given $f: X \to Y$ (C^∞),

$f_*, f_! : Sh(X) \to Sh(Y)$ direct image [resp. direct image w./compact support functor]

$f^{-1}, f^*, f^! : Sh(Y) \to Sh(X)$ inverse image functors.

(i) $(f_*\mathcal{F})(U) := \mathcal{F}(f^{-1}(U))$

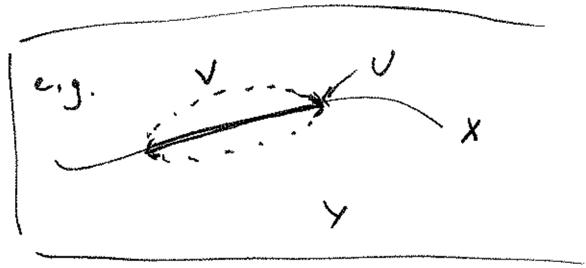
- if $\begin{cases} X = \text{pt. } p \\ f \text{ its inclusion in } Y \end{cases}$, then $(f_*\mathcal{O}_p)(U) = \begin{cases} \mathbb{C} & p \in U \\ 0 & p \notin U \end{cases}$

Note: $\mathcal{F} \in Sh(X)$
$\mathcal{G} \in Sh(Y)$

- if $f : \underset{\substack{\text{punctured}\\ \text{disk}\\ \text{about}\\ 0}}{\Delta^*} \hookrightarrow \underset{\substack{\text{usual}\\ \text{disk}\\ \text{about } 0\\ \text{in } \mathbb{C}}}{\Delta}$, then $(\mathcal{O}_\Delta)_0$ = convergent power series
  $(f_* \mathcal{O}_{\Delta^*})_0$ = convergent Laurent series.

(ii) $f^{-1}\mathcal{G}$:= sheaf associated to presheaf

$$U \longmapsto \varinjlim_{\substack{V \supseteq f(U)\\ V \in Op(Y)}} \mathcal{G}(V)$$



The stalks are just $(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}$. For sheaves of $\mathcal{O}$-modules, use instead

$f^*\mathcal{G} := f^{-1}\mathcal{G} \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X$. Remark that if $X \hookrightarrow Y$ is an open subset, then $f^{-1}\mathcal{G} = f^*\mathcal{G} =: $ "$\underline{\mathcal{G}|_X}$".

(iii) $(f_! \mathcal{F})(U) := \left\{ \sigma \in \mathcal{F}(f^{-1}(U)) \;\middle|\; \underbrace{f|_{supp(\sigma)} \text{ is proper}}_{} \right\}$

That is, the fibres of $\begin{matrix} supp(\sigma) \\ \downarrow f| \\ Y \end{matrix}$ are compact.

This is easiest to think about when (again) $X \hookrightarrow Y$ is an open subset. e.g., $f_! A^p_X$ = sheaves of p-forms compactly supported away from $Y \setminus X$.

(iv) $f^!\mathcal{G} := f^*\tilde{\mathcal{G}}$, where

assume f is an immersion of locally closed subspace $X \subseteq Y$

$$\tilde{\mathcal{G}}(U) := \{ \sigma \in \mathcal{G}(U) \mid supp(\sigma) \subseteq X \}.$$

An important example related to (i) is the following: if $V \overset{\iota}{\hookrightarrow} M$ is a closed $\mathbb{C}$-submanifold of a $\mathbb{C}$-mfld.,

$$\mathcal{I}_V := \underbrace{\ker\left(\mathcal{O}_M \to \iota_* \mathcal{O}_V\right)}_{(= \text{holo. fns. vanishing on } V)} \quad \text{is the } \underline{\text{ideal sheaf}}.$$

If V is of codim. 1, then we write instead $\mathcal{O}_M(-V)$.

So $\quad 0 \to \mathcal{O}_M(-V) \to \mathcal{O}_M \to \iota_*\mathcal{O}_V \to 0 \quad$ is short exact.

Example 8: Continue to assume $V \overset{i}{\hookrightarrow} M$ as above (codim. 1).

To get a broader perspective on the last example, consider

(i) the sheaves

$$\mathcal{O}_M(aV)(U) := \{ h \in \mathcal{M}(U) \cap \mathcal{O}(U \setminus V) \mid \mathrm{ord}_V(h) \geq -a \}$$

sheaf of sections

and

(ii) the line bundles $\mathcal{O}_M(aV)$, defined by taking a covering $\{\mathcal{U}_\alpha\}$ of M in which $\mathcal{U}_\alpha \cap V = \{f_\alpha = 0\}$ and $\Phi_{\alpha\beta} = \left(\frac{f_\alpha}{f_\beta}\right)^a$. Then sections consist of $\{g_\alpha\} \in \mathcal{O}(\mathcal{U}_\alpha)$ s.t.

$$g_\beta \left(\frac{f_\alpha}{f_\beta}\right)^a = g_\alpha \, , \qquad \text{i.e.}$$

$$\text{(F.1)} \qquad \boxed{\frac{g_\beta}{f_\beta^a} = \frac{g_\alpha}{f_\alpha^a}} \, ,$$

patching together to give an h as in (i).

(So taking sheaves of sections goes from (i) to (ii), giving a concrete example of the equivalence between sheaves of free $\mathcal{O}$-modules and [sheaves of sections of] vector bundles.)

A special case: writing $H = \{Z_0 = 0\} \subseteq \mathbb{P}^n$,

$$\mathcal{O}_{\mathbb{P}^n}(a) = \mathcal{O}_{\mathbb{P}^n}(aH).$$

Example 9: Čech resolution (this will be important for sheaf cohomology)

Let $\mathcal{F} \in Sh(M)$, $\{\mathcal{U}_i\}$ cover, $j_I : U_I (= \mathcal{U}_{i_1} \cap \cdots \cap \mathcal{U}_{i_{|I|}}) \hookrightarrow M$

$$\mathcal{F}_I := (j_I)_* \mathcal{F}|_{\mathcal{U}_I}$$

$$\mathcal{F}^{(k)} := \bigoplus_{|I| = k+1} \mathcal{F}_I .$$

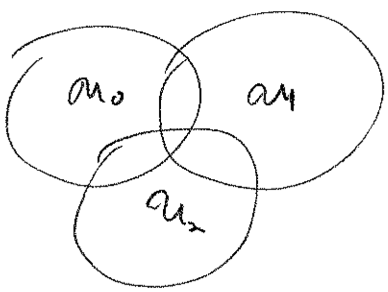
Define $d : \mathcal{F}^{(k)} \to \mathcal{F}^{(k+1)}$ by (on $\mathcal{U}$)

$$\underset{\mathcal{F}^{(k)}(\mathcal{U})}{\{\sigma_I\}} \longmapsto \underset{\mathcal{F}^{(k+1)}(\mathcal{U})}{\{(\delta\sigma)_J\}} , \quad \text{where}$$

$$(\delta\sigma)_{j_0, \ldots, j_{k+1}} = \sum_i (-1)^i \sigma_{j_0, \ldots, \hat{j_i}, \ldots, j_{k+1}}\Big|_{\mathcal{U} \cap \mathcal{U}_{j_0, \ldots, j_{k+1}}} , \quad j_0 < \cdots < j_{k+1}$$

Then $d \circ d = 0$ and $0 \to \mathcal{F} \underset{(\oplus|_{\mathcal{U}_i})}{\to} \mathcal{F}^{(0)} \xrightarrow{\delta} \mathcal{F}^{(1)} \xrightarrow{\delta} \mathcal{F}^{(2)} \xrightarrow{\delta} \cdots$ is exact.

e.g., if the cover looks like this (and $\mathcal{U}$ their union):



then

	matrices	rank
$\mathcal{F}(\mathcal{U}) = \mathcal{F}(\mathcal{U}_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2)$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	1
$\mathcal{F}^{(0)}(\mathcal{U}) = \mathcal{F}(\mathcal{U}_0) \oplus \mathcal{F}(\mathcal{U}_1) \oplus \mathcal{F}(\mathcal{U}_2)$	$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$	2
$\mathcal{F}^{(1)}(\mathcal{U}) = \mathcal{F}(\mathcal{U}_{01}) \oplus \mathcal{F}(\mathcal{U}_{12}) \oplus \mathcal{F}(\mathcal{U}_{02})$	$\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$	1
$\mathcal{F}^{(2)}(\mathcal{U}) = \mathcal{F}(\mathcal{U}_{012})$		

which confirms exactness.

Finally, one has the important functor $\Gamma_M : \underline{Sh(M)} \to \underline{Ab}$ defined (or just Γ)

by $\Gamma_M \mathcal{F} := \mathcal{F}(M)$ or "$\Gamma(M, \mathcal{F})$". So for example $\boxed{\Gamma(\mathbb{P}^n, \mathcal{O}(d)) \cong S^d_{n+1}}$

Right-derived functors (of left-exact functors)

An *abelian category* $\mathcal{C}$ ($= \underline{Ab}$, $\underline{Vec_k}$, $Sh(M)$, etc.) is one in which morphisms & objects can be added and there exist well-behaved kernels & cokernels for each morphism.*

(As usual, a morphism ϕ is $\begin{cases} \text{injective (mono)} \\ \text{surjective (epi)} \\ \text{isomorphism} \end{cases} \iff \begin{cases} \ker\phi = 0 \\ \operatorname{cok}\phi = 0 \\ \text{both} \end{cases}$.)

This is necessary in order, given a complex in $\mathcal{C}$

$$(A^\bullet, d) = \{0 \to A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \cdots\} \quad (d\circ d = 0),$$

for the (k^{th} degree) homology $H^k(A^\bullet) := \text{"}\frac{\ker(d^k)}{\operatorname{im}(d^{k-1})}\text{"} = \operatorname{cok}(A^{k-1} \xrightarrow{d^{k-1}} \ker d^k)$ to make sense. $(A^\bullet, d)$ is exact $\iff$ all $H^k(A^\bullet) = \{0\}$.

Two key results from homological algebra:

The 5-lemma: Given the commuting diagram w/ exact rows

$$\begin{array}{ccccccccc} X_1 & \to & X_2 & \to & X_3 & \to & X_4 & \to & X_5 \\ \downarrow & & \downarrow\cong & & \downarrow f_3 & & \downarrow\cong & & \int \\ Y_1 & \to & Y_2 & \to & Y_3 & \to & Y_4 & \to & Y_5 \end{array}, \quad f_3 \text{ is an } \cong.$$

do inj.

Ex/ prove the surjectivity part

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* a little less vaguely:

- morphisms $\operatorname{Hom}_{\mathcal{C}}(X,Y) \in \underline{Ab}$ (compositions biadditive; functors give homomorphisms of these)
- a zero object 0, with $\operatorname{Hom}_{\mathcal{C}}(0,0) = \{0\} \in \underline{Ab}$
- each $\phi \in \operatorname{Hom}_{\mathcal{C}}(X,Y)$ has kernel ($\ker\phi \to X$ s.t. $\psi \in \operatorname{Hom}(M,X)$ factors thru $\ker\phi$ $\iff \phi\circ\psi = 0$) (object together with morphism)
  & cokernel ($Y \to \operatorname{cok}\phi$ s.t. $\psi \in \operatorname{Hom}(Y,M)$ factors thru $\operatorname{cok}\phi$ $\iff \psi\circ\phi = 0$)
- direct sums exist (functors respect these)

The Snake Lemma: Given the commuting diagram w/ exact rows

$$\begin{array}{ccccccccc} 0 & \to & X_1 & \xrightarrow{g_1} & X_2 & \xrightarrow{g_2} & X_3 & \to & 0 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\ 0 & \to & Y_1 & \xrightarrow{h_1} & Y_2 & \xrightarrow{h_2} & Y_3 & \to & 0 \end{array}$$

"short-exact sequence" (sometimes the 0's on either end will be omitted [but understood] when this term is invoked)

we have a long-exact sequence

$$0 \to \ker f_1 \to \ker f_2 \to \ker f_3 \xrightarrow{\delta} \mathrm{cok}\, f_1 \to \mathrm{cok}\, f_2 \to \mathrm{cok}\, f_3 \to 0$$

where the connecting homomorphism δ is defined by "$h_1^{-1} \circ f_2 \circ g_2^{-1}$".

Definition 5:
An object $I \in \mathcal{C}$ is injective $\Leftrightarrow$ every diagram

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & \swarrow & \\ I & & \end{array}$$

can be completed. //

Assume $\mathcal{C}$ has enough injectives, i.e. each object embeds in an injective object.

Lemma 1: Every $A \in \mathcal{C}$ has an injective resolution $A \to I^\cdot$, i.e.

$\begin{cases} 0 \to A \hookrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots \text{ is exact} \\ \text{and } I^k\text{'s are injective.} \end{cases}$

Proof: $A \hookrightarrow I^0$, $\mathrm{cok}(\iota) \hookrightarrow I^1$, then

$$\begin{array}{ccccccc} I^0 & \xrightarrow{d^0} & & I^1 & \xrightarrow{d^1} & & I^2 \\ & \searrow & \nearrow & & \searrow & \nearrow & & \searrow \\ & & \mathrm{cok}(\iota) & & & \mathrm{cok}(d^0) & & \text{etc.} \end{array}$$

$\square$

A morphism of complexes $\phi^\cdot : A^\cdot \to B^\cdot$ (a collection $\phi^i : A^i \to B^i$ s.t. $d^i \circ \phi^i = \phi^{i+1} \circ d^i$) induces a morphism $H^k(\phi^\cdot) : H^k(A^\cdot) \to H^k(B^\cdot)$ ($\forall k$). (If this is $\cong$ for each k, ϕ is a quasi-isomorphism and we write $A^\cdot \simeq B^\cdot$.)

By the Snake Lemma, a short-exact sequence $A^\cdot \hookrightarrow B^\cdot \twoheadrightarrow C^\cdot$ of complexes (i.e. each $A^i \hookrightarrow B^i \twoheadrightarrow C^i$ is short-exact) induces a long-exact sequence

$$0 \to H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet) \xrightarrow{\delta} H^1(A^\bullet) \to H^1(B^\bullet) \to H^1(C^\bullet) \xrightarrow{\delta} H^2(A^\bullet) \to \cdots$$

Definition 6: $\phi^\bullet, \psi^\bullet : A^\bullet \to B^\bullet$ are homotopic "$\sim$" $\iff$

$\exists\, \kappa^\bullet : A^\bullet \to B[-1]^\bullet (= B^{\bullet-1})$ s.t. $\phi^\bullet - \psi^\bullet = d\circ\kappa + \kappa\circ d$. //

An easy exercise (it's just the proof of the d-Poincaré lemma) is

Lemma 2: $\phi^\bullet \sim \psi^\bullet \implies H^k(\phi^\bullet) = H^k(\psi^\bullet)$

Definition 7: A functor $F : \mathcal{C} \to \mathcal{D}$ (some other abelian category) is left-exact $\iff$ for any short-exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ in $\mathcal{C}$, $0 \to F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C)$ is exact: i.e. $F(\alpha)$ is injective, and $\operatorname{im} F(\alpha) = \ker F(\beta)$. [NOTE: it's enough that $0 \to A \to B \to C$ be exact.]

Example 10: (i) $\operatorname{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to Ab$ (exercise)

(ii) $\Gamma : Sh(M) \to Ab$

[Proof: given a s.e.s. of sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$, we get a diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{F}(M) & \xrightarrow{\eta} & \mathcal{G}(M) & \xrightarrow{\mu} & \mathcal{H}(M) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bigoplus_p \mathcal{F}_p & \xrightarrow{\oplus \eta_p} & \bigoplus_p \mathcal{G}_p & \xrightarrow{\oplus \mu_p} & \bigoplus_p \mathcal{H}_p & \to & 0 \end{array}$$

← not necessarily exact (top row)
← injective by vanishing property of sheaves (vertical maps)
← exact by definition (bottom row)

Injectivity of η is clear. Given $g \in \mathcal{G}(M)$, with $\mu(g) = 0$, we have $\{g_p\} \in \oplus \mathcal{G}_p$ with $\mu_p(g_p)$ all 0 hence $(\forall p)$ $g_p = \eta_p(f_p)$ for some $f_p \in \mathcal{F}_p$. Since g_p's agree on overlaps, and η_p are injective, so do the f_p's; by glueing we get $f \in \mathcal{F}(M)$ mapping to g. $\square$]

Assume F is left-exact.

Definition 8: The right-derived functors of F are defined by

$$\underset{\in\mathcal{C}}{(R^iF)(A)} := \underbrace{H^i(F(I^\bullet))}_{\in \mathcal{D}}, \quad A \hookrightarrow I^\bullet \text{ any injective resolution.}$$

So clearly

(F.2) $\boxed{(R^0F)(A) = F(A)}$

[Pf: $0 \to A \hookrightarrow I^0 \to I^1$ exact

⇓ F left-exact

$0 \to F(A) \to F(I^0) \to F(I^1)$ still exact (at $F(A)$, $F(I^0)$),

so $F(A) = \ker(F(I^0) \to F(I^1)) = H^1(F(I^\bullet))$.]

but are the others $(i>0)$ well-defined?

Lemma 3: Let $A \to I^\bullet$, $B \to J^\bullet$ be injective resolutions, $f \in \operatorname{Hom}_{\mathcal{C}}(A,B)$. Then $\exists$ morphism of complexes $\phi : I^\bullet \to J^\bullet$ with $\phi^0|_A = f$, unique up to homotopy.

(Sketch)
Proof of Lemma:

using fact that the J^i are injective

(continue in this fashion to get all ϕ^i)

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] \arrow[dr] & I^0 \arrow[r] \arrow[d, dashed, "\phi^0"] & \cdots \\ B \arrow[r, hook] & J^0 \arrow[r] & {} \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, hook, "\iota"] \arrow[d] & I^0 \arrow[r] \arrow[d] \arrow[dr, two heads] & I^1 \arrow[r] \arrow[d, dashed, "\phi^1"] & \cdots \\ B \arrow[r, hook] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots \end{tikzcd}$$
$\mathrm{cok}(\iota)$, 0

Now given $\phi^\cdot, \psi^\cdot$:

$$\begin{tikzcd} A \arrow[r, hook, "\iota"] \arrow[d, "0"'] & I^0 \arrow[rr] \arrow[d, "\phi^0-\psi^0"] \arrow[r, two heads] & \mathrm{cok}(\iota) \arrow[r, hook] & I^1 \arrow[d, "\phi^1-\psi^1"] \arrow[lld, dashed, "K^1"'] \\ B \arrow[r, hook] & J^0 \arrow[rr] & & J^1 \end{tikzcd}$$
etc.

□

Applications: (i) Given $A \rightarrow I^\cdot$, $A \rightarrow J^\cdot$ inj. resolutions, we have $\begin{cases} \xi^\cdot : I^\cdot \rightarrow J^\cdot \\ \psi^\cdot : J^\cdot \rightarrow I^\cdot \end{cases}$ $(\xi^0|_A = \mathrm{id} = \psi^0|_A)$

and $\begin{cases} \xi^\cdot \circ \psi^\cdot \sim \mathrm{id}_J \\ \psi^\cdot \circ \xi^\cdot \sim \mathrm{id}_I \end{cases}$, by the Lemma 3. Applying F,

$$\begin{cases} F(\xi^\cdot) \circ F(\psi^\cdot) \sim \mathrm{id}_{F(J)} \\ F(\psi^\cdot) \circ F(\xi^\cdot) \sim \mathrm{id}_{F(I)} \end{cases} \Rightarrow \begin{cases} H(F(\xi)) \circ H(F(\psi)) = \mathrm{id} \text{ on each } H^k(F(J)) \\ H(F(\psi)) \circ H(F(\xi)) = \mathrm{id} \text{ on each } H^k(F(I)) \end{cases}$$

$$\Rightarrow H^k(F(J)) \cong H^k(F(I)) \quad (\forall k).$$

(ii) We have (taking the trivial resolution $I \rightarrow I$) for I injective

(F.3) $$\boxed{R^i F(I) = 0 \quad \text{for } i > 0}.$$

(iii) The lemma also produces morphisms (well-defined b/c of ! up to htpy of ϕ)

(F.4) $$\boxed{R^i F(f) := H^i(F(\phi^\cdot)) : R^i F(A) \rightarrow R^i F(B)}$$

Given a short-exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$, F preserves injectivity on the left (and exactness in the middle) but not surjectivity on the right.

Proposition 1: The R^iF show how to continue to get an exact sequence (in $\mathcal{D}$):

$$(F.5) \qquad \boxed{0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to \cdots}$$

Proof: Choose $I^\cdot$, $K^\cdot$ resolutions:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, dotted] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r, dashed] & I^\cdot \arrow[r, dashed] & J^\cdot \arrow[r, dashed] & K^\cdot \arrow[r, dashed] & 0 \end{tikzcd}$$

want to construct so that $B \to J^\cdot$ is injective resolution and $I^\cdot \to J^\cdot \to K^\cdot$ is s.e.s. of complexes

Idea: take each $J^k = I^k \oplus K^k$. The first step is

$$\begin{tikzcd} A \arrow[r, hook, "\mathbb{I}"] \arrow[d, hook, "\iota"'] & B \arrow[r, two heads, "\mathbb{J}"] \arrow[d, dotted, "\alpha\oplus\beta"] \arrow[dl, phantom, "\alpha"] \arrow[dr, "\beta"] & C \arrow[d, hook, "j"] \\ I^0 \arrow[r, hook] & I^0\oplus K^0 \arrow[r, shift left] & K^0 \arrow[l, shift left] \end{tikzcd}$$

To see $\alpha\oplus\beta$ injective, $(\alpha\oplus\beta)(b)=0 \Rightarrow \beta(b)=0 \Rightarrow j(\mathbb{J}(b))=0$

$\Rightarrow \mathbb{J}(b)=0 \Rightarrow b=\mathbb{I}(a) \underset{\alpha(b)=0}{\Longrightarrow} \iota(a)=0 \Rightarrow a=0 \Rightarrow b=0.$

Ex/ Now consider $\operatorname{coke}(\alpha\oplus\beta)$ and embed into $I^1\oplus K^1$, etc.
— finish the proof.

$\square$

Proposition 2: The $(R^iF)(-)$ as defined in Defn. 8 are the unique "things" which satisfy (F.2), (F.3), and (F.5).

(they're "only almost" actual functors)

Proof: Forget the formula for $(R^kF)(A)$ but assume the 3 properties.

From $A \overset{\iota}{\hookrightarrow} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots$ we have the short-exact sequences

$$\left.\begin{array}{l} A \overset{\iota}{\hookrightarrow} I^0 \twoheadrightarrow \mathrm{cok}(\iota) \\ \mathrm{cok}(\iota) \hookrightarrow I^1 \twoheadrightarrow \mathrm{cok}(d^0) \\ \vdots \\ \mathrm{cok}(d^{k-2}) \hookrightarrow I^k \twoheadrightarrow \mathrm{cok}(d^{k-1}) \\ (\text{note } \mathrm{cok}(d^{k-2}) \cong \mathrm{im}(d^{k-1}) \cong \ker(d^k)) \end{array}\right\} \overset{(F.3)\,(F.5)}{\Longrightarrow} \begin{array}{l} (R^kF)(A) \cong (R^{k-1}F)(\mathrm{cok}(\iota)) \\ \cong (R^{k-2}F)(\mathrm{cok}(d^0)) \\ \vdots \\ \cong (R^1F)(\mathrm{cok}\, d^{k-3}) \\ \cong \mathrm{cok}\{I^{k-1} \to \ker d^k\} \\ = H^k(I^\cdot). \end{array}$$

□

Aside: For purposes of sheaf cohomology, it will be useful to know that we can take resolutions by a broader class of objects:

Definition 9: $\mathcal{L} \in \mathcal{C}$ is F-acyclic $\iff (R^iF)(\mathcal{L}) = 0 \quad \forall i > 0$.

Proposition 3: If $A \to \mathcal{L}^\cdot$ is an F-acyclic resolution then $(R^iF)(A) \cong H^i(F(\mathcal{L}^\cdot))$.

Proof: Imitate the last proof, using $(R^iF)(\mathcal{L}^j) = 0 \quad (i>0)$. □

Finally, a couple of examples (some light at the end of the tunnel!):

(referring to Example 10 above)

Example 11: (i) $R^k\mathrm{Hom}_{\mathcal{C}}(X, -) =: \mathrm{Ext}^k_{\mathcal{C}}(X, -)$ ("ext-groups" or extension)

(ii) $(R^k\Gamma_M)(-) =: H^k(M, -)$ (sheaf cohomology)

de Rham + Dolbeault cohomologies as sheaf cohomology

The work pays off:

Theorem 1: (i) $H^k\left(M, \begin{Bmatrix}\mathbb{R}\\ \mathbb{C}\end{Bmatrix}\right) \cong H^k_{dR}\left(M, \begin{Bmatrix}\mathbb{R}\\ \mathbb{C}\end{Bmatrix}\right)$

(ii) $H^q(M, \mathcal{O}(E)) \cong H^q_{\bar\partial}(M, E)$

(iii) $H^q(M, \Omega^p) \cong H^{p,q}_{\bar\partial}(M, \mathbb{C})$ i.e. $\mathcal{O}(\Lambda^p T_M^{\vee(1,0)})$

(ii), (iii) for M $\mathbb{C}$-manifold

But we'll need a few lemmas. Let $\mathcal{F} \in Sh(M)$.

Definition 10: (a) $\mathcal{F}$ *flasque* $\iff$ $\forall$ pairs $V \subset U$ ($\in Op(M)$), $\mathcal{F}(U) \overset{\rho_{VU}}{\twoheadrightarrow} \mathcal{F}(V)$ surjects.

(b) $\mathcal{F}$ *fine* $\iff$ $\forall$ covers $\{\mathcal{U}_\alpha\}$, $\exists$ endomorphisms $\{\eta_\alpha\}$ of $\mathcal{F}$ supported on $\mathcal{U}_\alpha$, with $\sum \eta_\alpha \equiv 1$.

Example 12: (a) $\mathcal{F}_{God} := \prod_{p\in M} \iota_{p*}\mathcal{F}_p$ $(\iota_p : \{p\} \hookrightarrow M)$ are *flasque*

(b) sheaves of C^∞-modules are *fine* (NOT flasque!)

Lemma 4: Given short-exact $0 \to \mathcal{F} \overset{\alpha}{\hookrightarrow} \mathcal{G} \overset{\beta}{\twoheadrightarrow} \mathcal{H} \to 0$, $\mathcal{G}, \mathcal{F}$ flasque $\Rightarrow \mathcal{H}$ flasque.

Lemma 5: flasque $\Rightarrow$ Γ_M-acyclic

Lemma 6: flasque resolutions exist (canonically)

Lemma 7: fine $\Rightarrow$ Γ_M-acyclic

Pf (Lemma 4): Claim: $\mathcal{G}(U) \twoheadrightarrow \mathcal{H}(U)$ $\forall\, U \in Op(M)$.

Idea of pf.: if $\sigma \in \mathcal{H}(U)$, $U = V \cup W$, $\begin{cases} \sigma|_V = \beta(\tau_V) \\ \sigma|_W = \beta(\tau_W) \end{cases}$, then

easy to prove in general from this

$\tau_V|_{VW} - \tau_W|_{VW} = \alpha(\chi_{VW})$. $\mathcal{F}$ flasque $\Rightarrow \exists\, \chi_V|_{VW} = \chi_{VW}$

$\Rightarrow (\tau_V - \alpha(\chi_V))|_{VW} = \tau_W|_{VW} \Rightarrow$ get τ on U with $\beta(\tau) = \sigma$. //

Finish by noting that in the diagram

$$\begin{tikzcd} \mathcal{H}(U) \arrow[r] & \mathcal{H}(V) \\ \mathcal{G}(U) \arrow[u, two heads, "\text{Claim}"] \arrow[r, two heads, "\mathcal{G}\text{ flasque}"'] & \mathcal{G}(V) \arrow[u, two heads, "\text{Claim}"'] \end{tikzcd}$$

this arrow must surject. □

Pf (Lemma 5): $\mathcal{F}_p \hookrightarrow \mathcal{I}_p$ (inj.) $\Rightarrow$

$$0 \to \underset{\text{flasque}}{\mathcal{F}} \hookrightarrow \underset{\substack{\text{inj.} \\ +\text{flasque}}}{\overset{\prod_{p\in M} \mathcal{I}_p}{\overset{\shortparallel}{\mathcal{I}}}} \to \underset{\text{flasque (by Lemma 4)}}{\overset{\text{cok}}{\overset{\shortparallel}{\mathcal{H}}}} \to 0$$

$\Rightarrow$ $0 \to \Gamma(M,\mathcal{F}) \to \Gamma(M,\mathcal{I}) \to \Gamma(M,\mathcal{H}) \to H^1(M,\mathcal{F}) \to \cancel{H^1(M,\mathcal{I})}_{0} \to H^1(M,\mathcal{H}) \xrightarrow{\cong} H^2(M,\mathcal{F})$

$\to \cancel{H^2(M,\mathcal{I})}_{0} \to H^2(M,\mathcal{H}) \xrightarrow{\cong} H^3(M,\mathcal{F}) \to \cancel{\cdots}_{0}$

$\Rightarrow$ $H^{k+1}(M,\mathcal{F}) \cong H^k(M,\mathcal{H})$ for $k \geq 1$.

Assuming inductively that flasque sheaves have $H^i = 0$ for $(0 \neq)\, i \leq k$, we get this for $i \leq k+1$. For getting the base case $k=1$, use the Claim above. $(\mathcal{I}(M) \twoheadrightarrow \mathcal{H}(M))$ □

Pf (Lemma 6):

$$\mathcal{F} \to \overbrace{\mathcal{F}_{God} \dashrightarrow (\mathcal{F}_{God}/\mathcal{F})_{God} \dashrightarrow}^{(\mathcal{G}\mathcal{F}^\bullet,\, \delta)}$$

$\mathcal{F}_{God} \twoheadrightarrow \mathcal{F}_{God}/\mathcal{F} \nearrow (\mathcal{F}_{God}/\mathcal{F})_{God} \twoheadrightarrow \ \nearrow$ etc. □

Pf (Lemma 7): $\mathcal{F} \hookrightarrow (\mathcal{G}\mathcal{F})^\bullet$ = flasque, $\underbrace{\text{fine sheaves}}$

& not just that, but δ commutes with the $\{\eta_\alpha\}$

Given $\begin{cases} f \in \ker(\mathcal{G}\mathcal{F}^k(M) \xrightarrow{\delta^k} \mathcal{G}\mathcal{F}^{k+1}(M)) \\ \{U_\alpha\} \subset M \text{ suff. small}, \end{cases}$ $f|_{U_\alpha} = \delta^{k-1}(g_\alpha)$.

Put $g := \sum \underset{(\text{part. of } 1)}{\eta_\alpha} g_\alpha \Rightarrow \delta^{k-1}(g) = \sum \eta_\alpha f|_{U_\alpha} = f$.

$\Rightarrow f \in \operatorname{im}(\delta^{k-1}) \Rightarrow H^k$ trivial. □

<u>Pf (Theorem 1)</u>: The RHS's are, by definition,

$$H^k\{A^\bullet(M), d\}, \quad H^q\{A^{0,\bullet}(M,E), \bar\partial\}, \quad H^q\{\underbrace{A^{p,\bullet}(M)}_{A^{0,\bullet}(\Lambda^p T_M^{*(1,0)})}, \bar\partial\}$$

so (ii) $\Rightarrow$ (iii).

So the point is that $\left\{ \begin{array}{l} \mathbb{R} \hookrightarrow (A^\bullet_{\mathbb{R}}, d) \\ \mathcal{O}(E) \hookrightarrow (A^{0,\bullet}(E), \bar\partial) \end{array} \right.$ are resolutions by C^∞ sheaves,

so done by Lemma 7 + Prop. 3. (Γ_M-acyclic $\Leftarrow$ fine) $\square$

<u>Corollary 1</u>: Let M = $\mathbb{C}$-n-manifold

(i) $H^k(M, \underset{\mathbb{C}}{\mathbb{R}}) = \{0\}$ for $k > 2\dim_{\mathbb{C}}(M)$ $\left(= \dim_{\mathbb{R}}(M)\right)$

(ii) $H^q(M, \mathcal{O}(E)) = \{0\}$ for $q > \dim_{\mathbb{C}}(M)$.

$\boxed{\text{Čech cohomology and sheaf cohomology}}$

(Historically, $\check{H}$ was the first coh. theory developed for sheaves.)

Let

$\mathfrak{U} = \{U_i\}$ op. cover of M, $\mathcal{F} \in Sh(M)$

$\mathcal{F} \hookrightarrow ((\check{C}_{\mathfrak{U}}\mathcal{F})^\bullet, \delta)$ associated Čech resolution,

$\check{H}^k(\mathfrak{U}, \mathcal{F}) := H^k\{\Gamma_M((\check{C}_{\mathfrak{U}}\mathcal{F})^\bullet), \delta\}$ the cohomology of its global sections.

<u>Definition 11</u>: <u>Čech cohomology</u> is $\check{H}^k(M, \mathcal{F}) := \varinjlim_{\mathfrak{U}} \check{H}^k(\mathfrak{U}, \mathcal{F})$.

Given $\mathcal{F} \hookrightarrow \mathcal{I}^\bullet$ injective resolution, we have

$$\begin{tikzcd} \mathcal{F} \arrow[r, hook] \arrow[d] & (\check{C}_{\mathfrak{U}}\mathcal{F})^\bullet \arrow[d, dashed, "\phi^\bullet_{\mathcal{F}}"] \\ \mathcal{F} \arrow[r, hook] & \mathcal{I}^\bullet \end{tikzcd}$$

(unique up to htpy.) morphism of complexes constructed as in proof of Lemma 3

Which induces a canonical morphism

$$\check{H}^q(\mathfrak{U}, \mathcal{F}) \xrightarrow[H^q(\phi^\bullet_{\mathcal{F}})]{} H^q(M, \mathcal{F}).$$

Theorem 2: The canonical morphism induces an isomorphism

$$\check{H}^q(M, \mathcal{F}) \xrightarrow{\cong} H^q(M, \mathcal{F}) \qquad (\text{in the limit}).$$

To see why this is true (we won't prove it in full), some preliminaries.

Lemma 8: (i) $\check{H}^0(\mathfrak{U}, \mathcal{F}) = \Gamma(M, \mathcal{F})$
(ii) $\check{H}^q(\mathfrak{U}, \mathcal{F}) = \{0\}$ for $\mathcal{F}$ flasque and $q > 0$.
$\Big\}$ $\mathfrak{U}$ arbitrary cover

Proof: (i) Ex/
(ii) the $(\check{C}_{\mathfrak{U}}\mathcal{F})^i$ are flasque (hence Γ_M-acyclic res. of $\mathcal{F}$) $\Rightarrow H^q\{\Gamma(\check{C}_{\mathfrak{U}}\mathcal{F}^\bullet), \delta\} \overset{\text{proposition 3}}{=} H^q(M, \mathcal{F}) \underset{\uparrow}{=} \{0\}$
(since $\mathcal{F}$ flasque $\Rightarrow \Gamma_M$-acyclic)

$\square$

Leray Theorem: If $H^i(U_I, \mathcal{F}) = \{0\}$ $(\forall I)$, then the $H^q(\phi)$ are $\cong$'s.

Proof: Embed $\mathcal{F} \hookrightarrow \underset{\text{flasque}}{\mathcal{G}} \twoheadrightarrow \mathcal{H}$ $\Rightarrow$
$\mathcal{H}$ = cok.

$$0 \to \mathcal{F}(U_I) \to \mathcal{G}(U_I) \to \mathcal{H}(U_I) \to \cancel{H^1(U_I, \mathcal{F})} \to \cdots \quad \text{0 by hypothesis}$$

From this + vanishing of $\underset{(\text{hypoth.})}{H^i(U_I, \mathcal{F})}$ & $\underset{(\text{flasque})}{H^i(U_I, \mathcal{G})}$ $\forall i > 0$, we see that $H^i(U_I, \mathcal{H}) = \{0\}$ $\forall i > 0$. Also, we see that

$$0 \to (\check{C}_{\mathfrak{U}}\mathcal{F})^i(M) \to (\check{C}_{\mathfrak{U}}\mathcal{G})^i(M) \to (\check{C}_{\mathfrak{U}}\mathcal{H})^i(M) \to 0$$

is an exact sequence of complexes, which $\Longrightarrow$

$(k > 0)$

$$\begin{tikzcd}
\cancel{\check{H}^k(\mathfrak{U}, \mathcal{G})} \arrow[r] & \check{H}^k(\mathfrak{U}, \mathcal{H}) \arrow[r, "\cong"] \arrow[d, "H^k(\phi_{\mathcal{H}})"] & \check{H}^{k+1}(\mathfrak{U}, \mathcal{F}) \arrow[r] \arrow[d, "H^{k+1}(\phi_{\mathcal{F}})"] & \cancel{\check{H}^{k+1}(\mathfrak{U}, \mathcal{G})} & (\check{C}\text{ech}) \\
\cancel{H^k(M, \mathcal{G})} \arrow[r] & H^k(M, \mathcal{H}) \arrow[r, "\cong"] & H^{k+1}(M, \mathcal{F}) \arrow[r] & \cancel{H^{k+1}(M, \mathcal{G})} & (\text{sheaf})
\end{tikzcd}$$

0 Lemma 8(ii) 0

0 ($\mathcal{G}$ flasque) 0

Now it's clear that we can induce on q, starting from base case $q = 0$ (Lemma 8(i)):
$H^{q-1}(\phi_{\mathcal{H}}) \cong \;\Rightarrow\; H^q(\phi_{\mathcal{F}}) \cong$, done.

$\square$

Ex / I've left out the argument to go from $q=0$ to $q=1$. Do this (using, say, the 5-lemma). //

Remark 4: (i) If F is a sheaf whose cohomology vanishes on contractible spaces, then I believe one can prove Theorem 2 from Leray by noting that good open covers — those in which all U_I are homeomorphic to a ball — are cofinal in the limit over open covers.

e.g. $\mathbb{R}, \mathcal{O}(E)$ by d-& $\bar\partial$-Poincaré

(ii) Let $F = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ (constant sheaves), M compact. Then applying Leray, wrt a good open cover, we see that $H^q(M, F)$ $\left(\cong \underline{\check{H}^q(\mathcal{U}, F)}\right)$ has finite rank. □

Singular + de Rham Cohomologies

We already introduced (defn. E-3) the complex of "singular chains"

$$(C_\bullet(M;\mathbb{Z}), \partial) \qquad [\partial \text{ \underline{decreases} degree}]$$

and the dual complex of "singular cochains"

$$(C^\bullet(M;\mathbb{Z}), \delta) \qquad [\delta = \partial^\vee \text{ \underline{increases} degree}]$$

whose (co)homologies define $H_q(M;\mathbb{Z})$ resp $H^q_{sing}(M;\mathbb{Z})$.

Let $\mathcal{C}^q :=$ sheafification of the presheaf $U \mapsto C^q(U;\mathbb{Z})$

$\mathcal{C}^q_\infty :=$ same but for differentiable (co)chains ;

both are flasque (obviously). Since balls are cofinal in the limit defining stalks,

$$\mathbb{Z} \hookrightarrow (\mathcal{C}^\bullet_{(\infty)}, \delta) \quad \text{is a resolution.}$$

Let $C^q(U;\mathbb{Z})_0 :=$ cochains which die on stalks ($\forall p \in U$); we have

$$\mathcal{C}^\bullet(M) = C^\bullet(M;\mathbb{Z}) \big/ C^\bullet(M;\mathbb{Z})_0 \xleftarrow[\text{standard result in algebraic topology}]{\substack{(\text{quasi-isom.}) \\ \simeq}} C^\bullet(M;\mathbb{Z}).$$

so

$$\begin{array}{ccc} H^q(\underbrace{\mathcal{C}^\bullet(M)}_{\text{flasque res.}}) & \xleftarrow{\cong} & H^q(C^\bullet(M;\mathbb{Z})) \\ \| & & \| \\ \underset{\text{sheaf coh.}}{H^q(M,\underline{\mathbb{Z}})} & & H^q_{sing}(M;\mathbb{Z}). \end{array}$$

Now there is an injective map of complexes of sheaves on M

$$A^\bullet_{\mathbb{R}} \xrightarrow{\int} \underbrace{C^\bullet_{\infty,\mathbb{R}}}_{(\text{i.e. } \otimes_{\mathbb{Z}} \mathbb{R})}$$

which sends $\omega \in A^k_{\mathbb{R}}(U)$ to the $\mathbb{R}$-valued functional on C^∞ k-chains $\int_{(\cdot)} \omega$. (Ex/ Why injective?) $\int$ is a map of complexes by Stokes: thinking of $\int$ as a pairing of forms and chains, d is adjoint to ∂ (hence equals δ). Considering the diagram

$$\begin{array}{ccc} \mathbb{R} & \hookrightarrow & A^\bullet_{\mathbb{R}} \ (\text{fine}) \\ \downarrow \| & & \downarrow \int \\ \mathbb{R} & \hookrightarrow & C^\bullet_{\infty,\mathbb{R}} \ (\text{flasque}) \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & C^\bullet_{\infty,\mathbb{R}}/A^\bullet_{\mathbb{R}} \end{array}$$

with exact columns and 1^{st} 2 rows exact, an easy diagram chase shows the 3rd row is exact too — hence a "resolution of 0". The long-exact cohomology sequence associated to the qth column is →

$$\cdots \to \underbrace{H^i(M, \mathcal{C}^q_{\infty,\mathbb{R}})}_{0} \to H^i(M, \mathcal{C}^q_{\infty,\mathbb{R}}/\mathcal{A}^q_{\mathbb{R}}) \to \underbrace{H^{i+1}(M, \mathcal{A}^q_{\mathbb{R}})}_{0} \to \cdots$$

$\Longrightarrow$ the $\mathcal{C}^q_{\infty,\mathbb{R}}/\mathcal{A}^q_{\mathbb{R}}$ are Γ_M-acyclic $\Longrightarrow$ all 3 resolutions are Γ_M-acyclic.

Applying Γ_M to the whole diagram, cohomology of rows computes (resp.) sheaf cohom. of $\mathbb{R}$, $\mathbb{R}$, and 0; but also, the first two are dR-cohom. & singular cohom. ($\mathbb{R}$-coeffs.). So we get a long-exact sequence (snake lemma)

$$\cdots \to \underbrace{H^{i-1}\left(\frac{\mathcal{C}^\bullet_{\infty,\mathbb{R}}}{\mathcal{A}^\bullet_{\mathbb{R}}}(M)\right)}_{0} \to H^i(\mathcal{A}^\bullet_{\mathbb{R}}(M)) \underset{\cong}{\overset{\int}{\to}} H^i(\mathcal{C}^\bullet_{\infty,\mathbb{R}}(M)) \to \underbrace{H^i\left(\frac{\mathcal{C}^\bullet_{\infty,\mathbb{R}}}{\mathcal{A}^\bullet_{\mathbb{R}}}(M)\right)}_{0} \to \cdots$$

0 (because it's computing the cohomology of 0!!)

implying the

<u>de Rham Theorem</u>: $\int$ induces isomorphism for all $k \geq 0$

$$H^k_{dR}(M, \mathbb{R}) \xrightarrow{\cong} H^k_{sing}(M; \mathbb{R}).$$

<u>Remark 5</u>: This means that $\int$ gives a perfect pairing (again, well-defined by Stokes)

$$H^k_{dR}(M, \mathbb{R}) \times H_k(M, \mathbb{R}) \to \mathbb{R}.$$

This says that the <u>periods</u> $\left\{ \int_{\gamma_i} \omega \right\}$ of a closed form record its

$\gamma_i \leftarrow$ basis of $\left(\frac{\text{cycles}}{\text{boundaries}}\right)$

de-Rham-cohomology class. Periods are absolutely central in Hodge theory and other parts of modern algebraic geometry + number theory.

There is a similar result on cohomology with compact support:

$$H^k_{dR,c}(M, \mathbb{R}) \times \underbrace{H_k(\overline{M}, \overline{M}\setminus M; \mathbb{R})}_{\text{relative homology}} \to \mathbb{R} \quad \text{is perfect},$$

which I'll make into an exercise.

$\boxed{\text{Integrality of } 1^{st} \text{ Chern class}}$ (An application of sheaf coh.)

Over a $\mathbb{C}$-manifold M,

$$H^1(M, \mathcal{O}^*) \cong \{\text{isom. classes of holomorphic line bundles}\}$$

$$H^1\left(M, \begin{cases}\mathbb{Q}^* \\ \mathbb{R}^* \\ \mathbb{C}^*\end{cases}\right) \cong \left\{\text{isom. classes of rank } 1 \begin{cases}\mathbb{Q}- \\ \mathbb{R}- \\ \mathbb{C}-\end{cases} \underline{\text{local systems}}\right\}$$

Local systems are the analogue of vector bundles but where the transition functions are constant. (Think of them as "locally constant sheaves" or "twisted coefficients".)

In both cases the idea of the proof is clear: think of (Čech) $\check{H}^1$, so that an element consists of $\Phi_{ij} \in \begin{cases}\mathcal{O}^*(U_{ij}) \\ \text{etc}\end{cases}$ which (to be δ-closed) must satisfy $\Phi_{ij}\Phi_{jk}\Phi_{ki} = 1$. But this is exactly the condition for a bundle/local system. (Cf. [Voisin] for more details).

Now let $L \xrightarrow{\pi} M$ be a holomorphic line bundle, and recall $c_1(L) \in H^2_{(dR)}(M, \mathbb{C})$ (Cf. Defn. E.4).

Theorem 3: $c_1(L) \in H^2(M, \mathbb{Z})$.

Proof: let $\mathcal{U}$ = good open cover of M, $\begin{cases}\check{C}^i(-) := \check{C}^i(\mathcal{U}, -) \\ \check{\delta} = \text{Čech differential}\end{cases}$

$\boxed{\text{Step 1}}$ Recall the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{e} \mathcal{O}^* \to 0$$

$e = \exp(2\pi i(\cdot))$

with long-exact cohomology sequence

$$\cdots \to H^1(M,\mathcal{O}) \to H^1(M,\mathcal{O}^*) \xrightarrow{\delta} H^2(M,\mathbb{Z}) \to \cdots$$

$$\text{as described above}\left\{\begin{array}{ccc} \Psi & & \Psi \\ [L] & \longmapsto & \delta[L] \end{array}\right.$$

We will prove $-c_1(L) = $ image of $\delta[L]$ in $H^2(M,\mathbb{C})$.

$\boxed{\text{Step 2}}$ Compute $\delta[L]$:

$$\begin{tikzcd}
 & & \left\{\frac{\log \Phi_{\alpha\beta}}{2\pi i}\right\} \arrow[rr, mapsto] & & {[L] = \{\Phi_{\alpha\beta}\}} & \\
0 \arrow[r] & \check{C}^1(\mathbb{Z}) \arrow[r] \arrow[d, "\check{\delta}"] & \check{C}^1(\mathcal{O}) \arrow[rr, "e"] \arrow[d, "\check{\delta}"] & & \check{C}^1(\mathcal{O}^*) \arrow[r] \arrow[d, "\check{\delta}"] & 0 \\
 & \check{C}^2(\mathbb{Z}) \arrow[r] & \check{C}^2(\mathcal{O}) \arrow[rr, "e"] & & \check{C}^2(\mathcal{O}^*) \arrow[r] & 0 \\
 & \delta[L] \arrow[r, mapsto] & \left\{\frac{\log \Phi_{\alpha\beta} - \log \Phi_{\alpha\gamma} + \log \Phi_{\beta\gamma}}{2\pi i}\right\} \arrow[rr, mapsto] & & 0 &
\end{tikzcd}$$

(by defn.)

given by same formula

$\boxed{\text{Step 3}}$ Inspired by the dR-resolution $\mathbb{C} \hookrightarrow A^\bullet$, we construct a double Čech–de Rham complex

$$\begin{tikzcd}
 & \uparrow \check{\delta} & \uparrow \check{\delta} & \uparrow \check{\delta} & \\
0 \arrow[r] & \check{C}^2(A^0) \arrow[r, "d"] \arrow[u] & \check{C}^2(A^1) \arrow[r, "d"] & \check{C}^2(A^2) \arrow[r, "d"] & {} \\
0 \arrow[r] & \check{C}^1(A^0) \arrow[r, "d"] \arrow[u, "\check{\delta}"] & \check{C}^1(A^1) \arrow[r, "d"] \arrow[u, "\check{\delta}"] & \check{C}^1(A^2) \arrow[r, "d"] \arrow[u, "\check{\delta}"] & {} \\
0 \arrow[r] & \check{C}^0(A^0) \arrow[r, "d"] \arrow[u, "\check{\delta}"] & \check{C}^0(A^1) \arrow[r, "d"] \arrow[u, "\check{\delta}"] & \check{C}^0(A^2) \arrow[r, "d"] \arrow[u, "\check{\delta}"] & {}
\end{tikzcd}$$

which has the associated simple complex

$$K^\bullet := \bigoplus_{i+j=\bullet} \check{C}^j(A^i) \qquad \text{with differential } D := d + (-1)^i \check{\delta}.$$

$\left(H^{j>0}(\mathcal{U}_I, A^k) = \{0\}\right)$

Now

- (a) A^i fine + Čech $\cong$ sheaf cohom. $\Longrightarrow$ $H^{j>0}(M, A^i) = \{0\}$

$\Longrightarrow$ columns are exact except at 0^{th} row

- (b) $\mathcal{U}$ good + d-Poincaré $\Longrightarrow$ rows exact except at 0^{th} column.

- (a)+(b) + diagram chase $\Longrightarrow$ any D-cocycle in K^ℓ can be moved by a D-coboundary into

$$\left(\left\{(\ker\check{\delta} \cap \ker d) \subset \check{C}^0(A^\ell)\right\} = \right) \{\ker d \subset A^\ell(M)\}$$

$$\text{or } \left(\left\{(\ker\check{\delta} \cap \ker d) \subset \check{C}^\ell(A^0)\right\} = \right) \{\ker\check{\delta} \subset \check{C}^\ell(\mathbb{C})\}$$

This yields explicit isomorphisms

$$\check{H}^\ell(\mathcal{U}, \mathbb{C}) \cong H^\ell(K^\bullet) \cong H^\ell_{dR}(M, \mathbb{C}),$$

especially between the end terms.

$\boxed{\text{Step 4}}$ Under the last isomorphism, we have

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow \check{\delta} & & \\
(\check{C}^2(A^0) \supseteq)\ \check{C}^2(\mathbb{C}) \ni \delta[L] & & & & \\
\uparrow \check{\delta} & & & & \\
\left\{\dfrac{\log \Phi_{\alpha\beta}}{2\pi i}\right\} & \overset{d}{\longmapsto} & \left\{\dfrac{\partial \log \Phi_{\alpha\beta}}{2\pi i}\right\} & & \\
 & & \uparrow \check{\delta} & & \\
 & & \left\{\dfrac{\partial \log \rho_\alpha}{2\pi i}\right\} & \overset{d}{\longmapsto} & \left\{\dfrac{-\partial\bar{\partial} \log \rho_\alpha}{2\pi i}\right\} \in A^2(M) \\
 & & & & \text{///} \qquad \cap \\
 & & & & -c_1(L), \qquad \check{C}^0(A^2)
\end{array}$$

$= d$ since $\Phi_{\alpha\beta}$ holo.

recall $\rho_\alpha = h(\sigma_\alpha, \sigma_\alpha)$ satisfied
$\rho_\beta/\rho_\alpha = |\Phi_{\alpha\beta}|^2 \Rightarrow$
$\log\rho_\beta - \log\rho_\alpha = \log\Phi_{\alpha\beta} + \log\bar{\Phi}_{\alpha\beta} \Rightarrow$
$\partial\log\rho_\beta - \partial\log\rho_\alpha = \partial\log\Phi_{\alpha\beta}$.

cf. Defn. E.4

$\square$

<u>Corollary 2</u>: A given compact $\mathbb{C}$-manifold M can be a projective a/g. variety ONLY IF M admits a Kähler metric whose Kähler class is integral. [compare Prop. E.1]

<u>Proof</u>: $M \subset \mathbb{P}^n \Rightarrow$ has Kähler form $\iota^*\omega_{FS} = \omega(\mathcal{O}_{\mathbb{P}^n}(1)|_M, h^*|_M)$

hence " class $[\iota^*\omega_{FS}] = c_1(\mathcal{O}_{\mathbb{P}^n}(1)|_M) \in H^2(M, \mathbb{Z})$. $\square$